Pre-Final Round 2023

DATE OF RELEASE: 19. OCTOBER 2023

Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend printing this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The eight problems are separated into three categories: 2x basic problems (A; four points), 2x advanced problems (B; six points), 2x special-creativity problems (C; eight points).
- You receive points for the correct solution as well as for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 36 points in total. You qualify for the final round if you reach at least 18 points (under 18 years) or 24 points (over 18 years).
- Please consider following notation that is used for the problems
  - $x, y \in \mathbb{R}$ denotes a real number, $n, k \in \mathbb{N}$ denotes a positive integer.
  - f, g, h denote functions. The domain and co-domain should follow from the context.
  - The *roots* of a function f are all x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
  - The natural logarithm of x is written as $\log(x)$.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/en/login`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no Word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sunday 22. October 2023, 23:59 UTC+0**.
- The results of the pre-final round will be announced on Monday 30. October 2023.

Good luck!

Problem A.1

You are given the following three functions:

$$f(x) = 3 + 3x \qquad g(x) = 2 + 2x \qquad h(x) = 5 - x$$

Find $a, b \in \mathbb{R}$ such that $h(x) < f(x)$ and $h(x) > g(x)$ for all x with $a < x < b$.

Problem A.2

Find the derivative $f'(x)$ of the following function with respect to x:

$$f(x) = (1 + x^2)^x$$

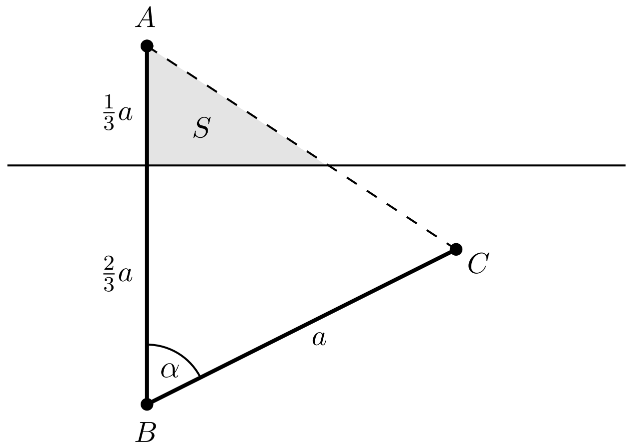
Problem B.1

Show that this infinite sum converges and determine its value:

$$\sum_{n=1}^{\infty} \frac{5^n + 5^{n+1}}{5^{2n+1}}$$

Problem B.2

A vertical line $\overline{AB}$ with length a is intersected by a horizontal line at $\frac{2}{3}a$. Another line $\overline{BC}$ with length a is rotated by an angle of α and attached to point B. Find an equation for the enclosed area $S(\alpha)$ between the horizontal line and the line $\overline{AC}$ for $\arccos(2/3) \leq \alpha \leq \pi$.



Problem C.1

For this problem, consider the following list of seven mysterious increasing positive integers:

n	$f(n)$
1	60
2	504
3	2160
4	18144
5	77760
6	653184
7	2799360

(a) Find at least four *properties*[1] that all seven numbers have in common.

(b) Explain the underlying pattern and give a function $f(n)$ to calculate the nth number
   i. with recursion.
   ii. without recursion.

(c) What are the numerical values of the 9th and 15th number?

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[1] *Properties* can be any nontrivial characteristic related to the digits, specific type of number, certain patterns, divisibility, etc. – be creative!

Problem C.2

This problem requires you to read following scientific article:

On Sándor's Inequality for the Riemann Zeta Function.
Alzer, H., Kwong, M. K. *Journal of Integer Sequences*, 26 (2023).
Link: https://cs.uwaterloo.ca/journals/JIS/VOL26/Alzer/alzer15.pdf

Use the content of the article to work on the problems (a-f) below:

(a) What is the numerical value of $\omega(100)$, $\zeta(2)$ and $\zeta_2(2)$?

(b) Show that $\dfrac{\zeta^3(5)}{\zeta(15)} < \sum_{n=1}^{\infty} \dfrac{6^{\omega(n)}}{n^5}$ by applying Theorem 1.

(c) Let $\gcd(m,n) = 1$. Explain why $\omega(mn) = \omega(m) + \omega(n)$ and $F_a(mn) = F_a(m)F_a(n)$.

(d) Prove that $\sum_{n=1}^{\infty} \dfrac{F_a(p^n q^n)}{(p^n q^n)^s} = a^2 \dfrac{(pq)^{-s}}{1-(pq)^{-s}}$ for two distinct prime numbers p and q.

(e) In equation (11), derive explicitly why the (left/first) equality holds true.

(f) Why do (11), (12), (13) imply that equation (4) is valid?