Pre-Final Round 2021

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Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend printing this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The 10 problems are separated into three categories: 4x basic problems (A; three points), 4x advanced problems (B; four points), 2x special-creativity problems (C; six points).
- You receive points for the correct solution as well as for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 40 points in total. You qualify for the final round if you reach at least 20 points (under 18 years) or 28 points (over 18 years).
- Please consider following notation that is used for the problems
  - $x, y \in \mathbb{R}$ denotes a real number, $n, k \in \mathbb{N}$ denotes a positive integer.
  - f, g, h denote functions. The domain and co-domain should follow from the context.
  - The *roots* of a function f are all x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
  - The natural logarithm of x is written as $\log(x)$.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

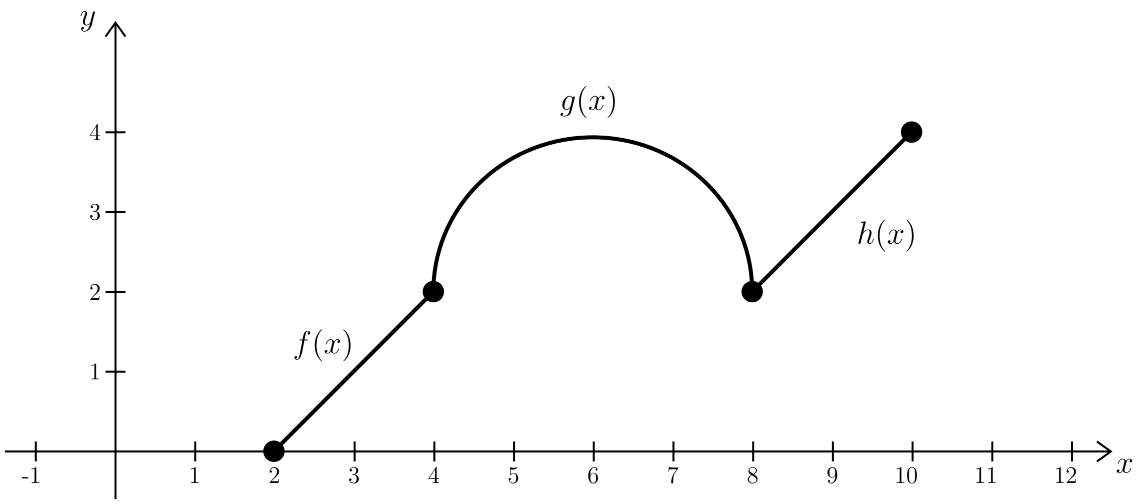
Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/en/login`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no Word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sunday 21. November 2021, 23:59 UTC+0**.
- The results of the pre-final round will be announced on Monday 29. November 2021.

Good luck!

Problem A.1

The graph below is made of three line segments:



The segments correspond to the following three functions:

$$f(x) = x - 2, \qquad g(x) = \sqrt{4 - (x-6)^2} + 2, \qquad h(x) = x - 6$$

Find the total length L of the graph between $x = 2$ and $x = 10$.

Solution:

$$\begin{aligned} L &= \sqrt{2^2 + f^2(4)} + \frac{4\pi}{2} + \sqrt{2^2 + (h(10) - h(8))^2} \\ &= \sqrt{2^2 + 2^2} + 2\pi + \sqrt{2^2 + 2^2} \\ &= 4\sqrt{2} + 2\pi \end{aligned}$$

Problem A.2

Let $f(x)$, $g(x)$ and $h(x)$ be the functions from Problem A.1. Find the derivative $\lambda'(x)$ of the following function with respect to x:

$$\lambda(x) = f(x) \cdot g(x) + f(x) \cdot h(x) - g(x) \cdot h(x)$$

Solution:

$$\begin{aligned}
\lambda'(x) &= f'g + fg' + f'h + fh' - g'h - gh' \\
&= g + fg' + h + f - g'h - g \\
&= (f - h)g' + h + f \\
&= \frac{4(x-6)}{\sqrt{4-(x-6)^2}} + 2x - 8
\end{aligned}$$

Problem A.3

The formula for calculating the sum of all natural integers from 1 to n is well-known:

$$S_n = 1 + 2 + 3 + ... + n = \frac{n^2 + n}{2}$$

Similary, we know about the formula for calculating the sum of all squares:

$$Q_n = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + ... + n \cdot n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Now, we reduce one of the two multipliers of each product by one to get the following sum:

$$M_n = 0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + (n-1) \cdot n$$

Find an explicit formula for calculating the sum M_n.

Solution:

$$M_n = \sum_{k=1}^{n} (k-1)k = \sum_{k=1}^{n} k^2 - k = Q_n - S_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - \frac{n^2}{2} - \frac{n}{2} = \frac{n^3 - n}{3}$$

Problem B.1

Find the smallest positive integer N that satisfies all of the following conditions:

- N is a square.
- N is a cube.
- N is an odd number.
- N is divisible by twelve prime numbers.

How many digits does this number N have?

Solution: Let $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ be the prime factorization of N with $p_i < p_{i+1}$.

- N is a square $\implies 2|\alpha_i$
- N is a cube $\implies 3|\alpha_i$; thus $\alpha_i = 2 \cdot 3 = 6$
- N is an odd number $\implies p_1 \geq 3$
- N is divisible by twelve prime numbers $\implies p_i \in \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41\}$

This gives:

$$N = (3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41)^6 = (152125131763605)^6$$

The number of digits can be determined by using logarithm:

$$\lfloor \log_{10}(N) \rfloor + 1 = \lfloor \log_{10}((152125131763605)^6) \rfloor + 1 = \lfloor 6 \cdot \log_{10}(152125131763605) \rfloor + 1 = 86$$

Note: Writing down the full number yields:

$$N = 12393837692914617514545128103985807911961532559239905437240736215373258683182067515625$$

Problem B.2

The following product can be expended into a power series with coefficients a_k:

$$\prod_{n=1}^{100}\left[\frac{x^3}{4^n}+\left(\frac{\pi}{2^n}\right)^2\right]=\sum_{k=0}^{\infty}a_k x^k$$

Find the coefficients a_k in front of the individual x^k terms for all $k\in\mathbb{N}$.

Solution: Applying the binomial theorem yields:

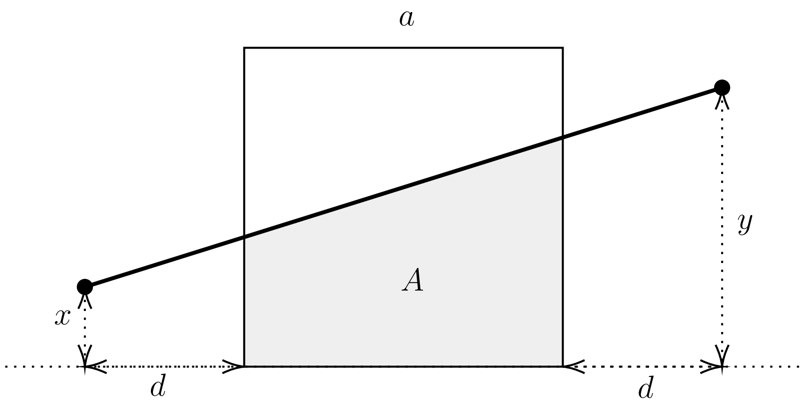
$$\begin{aligned}\prod_{n=1}^{100}\left[\frac{x^3}{4^n}+\left(\frac{\pi}{2^n}\right)^2\right]&=\prod_{n=1}^{100}\left[\frac{x^3}{4^n}+\frac{\pi^2}{4^n}\right]\\&=(x^3+\pi^2)^{100}\cdot\prod_{n=1}^{100}\frac{1}{4^n}\\&=\sum_{k=0}^{100}\binom{100}{k}(x^3)^k(\pi^2)^{100-k}\cdot\prod_{n=1}^{100}\frac{1}{4^n}\\&=\sum_{k=0}^{100}\binom{100}{k}x^{3k}\pi^{200-2k}\cdot\frac{1}{4^{(100+100^2)/2}}\\&=\sum_{k=0}^{100}\binom{100}{k}\frac{\pi^{200-2k}}{2^{10100}}\cdot x^{3k}\end{aligned}$$

Thus, we get for the coefficients:

$$a_k=\begin{cases}\binom{100}{k/3}\frac{\pi^{200-2k/3}}{2^{10100}} & ,\ 3\mid k\\ 0 & ,\ 3\nmid k \text{ or } k>300\end{cases}$$

Problem B.3

The drawing below shows a square with side a. A straight line intersects the square and encloses an area A. The heights x and y on the left and right side (in a distance d from the square) of the intersecting line can be varied. Assuming that $x \leq y$ and $x, y \leq a$, find an expression for the enclosed area $A(x, y)$ with respect to x and y.



Solution: The area A can be seperated into a rectanglular area and a triangular area:

$$\begin{aligned} A &= A_R + A_R \\ &= a \cdot (x + \Delta) + \frac{a}{2}([y - \Delta] - [x + \Delta]) \\ &= a \cdot (x + \Delta + \frac{y}{2} - \frac{x}{2} - \Delta) \\ &= a \cdot \frac{x + y}{2} \end{aligned}$$

Problem C.1

The well-known formula for calculating the sum S_n of the positive integers from 1 to n was already part of Problem A.3. For this problem, we consider the following *rollercoaster sum*:

$$S_n^{(2)} = 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 4 + ... + 1 \cdot (n-1) + 2 \cdot n$$

Here, we multiple the summands successively with 1, 2, 1, 2, 1, 2, ...

(a) Find an explicit formula to calculate this sum $S_N^{(2)}$. (Assume that n is a multiple of 2.)

Now, we consider the sum $S_n^{(3)}$:

$$S_n^{(3)} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + ... + 1 \cdot (n-2) + 2 \cdot (n-1) + 3 \cdot n$$

Here, we multiple the summands successively with 1, 2, 3, 1, 2, 3, ...

(b) Again, find an explicite formula for the sum $S_n^{(3)}$. (Assume that n is a multiple of 3.)

(c) Express $S_n^{(3)}$ in the form of

$$S_n^{(3)} = I \cdot S_{n/3} - Y \cdot n$$

where S_n is the formula from Problem A.3 and I, Y are rational constants.

(d) Find a formula for the general case of $S_n^{(m)}$. (That means we multiple the summands successively with 1, 2, 3, ..., m, 1, 2, 3, ..., m, ...; Assume that n is a multiple of m.)

(e) Now, express the general formula as

$$S_n^{(m)} = I_m \cdot S_{n/m} - Y_m \cdot n$$

and find explicit equations to calculate I_m and Y_m for a given m.

(f) Determine the growth behaviour by expressing I_m and Y_m with the *big O notation*.

Solution:

(a)

$$S_n^{(2)} = \sum_{k=1}^{n/2}(2k-1)+2(2k) = \sum_{k=1}^{n/2} 6k-1 = 6\left(\sum_{k=1}^{n/2} k\right) - \frac{1}{2}n = 6\left(\frac{n^2}{8}+\frac{n}{4}\right) - \frac{1}{2}n = \frac{3}{4}n^2 + n$$

(b)

$$S_n^{(3)} = \sum_{k=1}^{n/3}(3k-2)+2(3k-1)+3(3k) = \sum_{k=1}^{n/3} 18k-4 = 18\left(\sum_{k=1}^{n/3} k\right) - \frac{4}{3}n = 18\left(\frac{n^2}{18}+\frac{n}{6}\right) - \frac{4}{3}n = n^2 + \frac{5}{3}n$$

(c)

$$S_n^{(3)} = 18 \cdot S_{n/3} - \frac{4}{3}n$$

(d)

$$\begin{aligned}
S_n^{(m)} &= \sum_{k=1}^{n/m}\left(\sum_{i=1}^{m} i\cdot(mk-m+i)\right) = \sum_{k=1}^{n/m}\left(\sum_{i=1}^{m} imk - im + i^2\right) = \sum_{k=1}^{n/m}\left(m(k-1)\sum_{i=1}^{m} i + \sum_{i=1}^{m} i^2\right) \\
&= m\left(\sum_{i=1}^{m} i\right)\sum_{k=1}^{n/m}(k-1) + \sum_{k=1}^{n/m}\left(\sum_{i=1}^{m} i^2\right) = m\left(\sum_{i=1}^{m} i\right)\left(\sum_{k=1}^{n/m} k - \frac{n}{m}\right) + \frac{n}{m}\left(\sum_{i=1}^{m} i^2\right) \\
&= m\cdot\frac{m^2+m}{2}\cdot\left(\frac{n^2/m^2+n/m}{2} - \frac{n}{m}\right) + \frac{n}{m}\cdot\left(\frac{m^3}{3}+\frac{m^2}{2}+\frac{m}{6}\right) \\
&= \frac{m^3+m^2}{2}\cdot\left(\frac{n^2+nm}{2m^2} - \frac{2nm}{2m^2}\right) + \frac{n}{m}\cdot\left(\frac{2m^3+3m^2+m}{6}\right) \\
&= \frac{m^3+m^2}{2}\cdot\frac{n^2-nm}{2m^2} + n\cdot\frac{2m^2+3m+1}{6} \\
&= \frac{(m+1)(n^2-nm)}{4} + n\cdot\frac{2m^2+3m+1}{6}
\end{aligned}$$

(e)

$$\begin{aligned}
S_n^{(m)} &= m\left(\sum_{i=1}^{m} i\right)\left(S_{n/m} - \frac{n}{m}\right) + \frac{n}{m}\left(\sum_{i=1}^{m} i^2\right) \\
&= \frac{m^3+m^2}{2}\cdot S_{n/m} - n\cdot\frac{m^2+m}{2} + \frac{n}{m}\left(\sum_{i=1}^{m} i^2\right) \\
&= \frac{m^3+m^2}{2}\cdot S_{n/m} - n\left(\frac{m^2+m}{2} - \frac{m^2}{3} - \frac{m}{2} - \frac{1}{6}\right) \\
&= \frac{m^3+m^2}{2}\cdot S_{n/m} - n\left(\frac{3m^2+3m-2m^2-3m-1}{6}\right) \\
&= \frac{m^3+m^2}{2}\cdot S_{n/m} - n\left(\frac{m^2-1}{6}\right)
\end{aligned}$$

Thus, we have:

$$I_m = \frac{m^3 + m^2}{2}, \qquad Y_m = \frac{m^2 - 1}{6}$$

(f)

$$I_m = \mathcal{O}(m^3), \qquad Y_m = \mathcal{O}(m^2)$$

Problem C.2

This problem requires you to read following scientific article:

On upper bounds for the count of elite primes.
Just, Matthew. Integers Volume 20 (2021).
Link: http://math.colgate.edu/ integers/vol20

Use the content of the article to work on the problems (a-f) below:

(a) What are *quadratic residues* and *nonresidues*?

(b) Write down the definition of *elite primes* in mathematical terms.

(c) Prove that $\prod_{i=0}^{2t} F_i = 2^{2^{2t+1}} - 1$.

(d) What does the ϕ in the Brun-Titchmarsh inquality represent?

(e) Explicitly show that $\pi(x; 2^t, 1) \ll \frac{x}{2^t}$.

(f) What can you say about the upper bound of $E(x)$ for numbers of the form $3^{2^n} + 1$?

Solution:

(a)

→ n is a *quadratic residue* modulo m if it is congruent to a square, i.e.

$$\exists k \in \mathbb{N}: \ k^2 \equiv n \mod m$$

→ otherwise n is a *nonresidue*

(b)

$$p \text{ is elite prime} \iff \exists N \in \mathbb{N}: \ \forall n > N : \nexists k \in \mathbb{N} : F_n \equiv k^2 \mod p$$

(c)

With induction:

1. Check for $t = 1$: $F_0 \cdot F_1 \cdot F_2 = 255 = 2^{2^3} - 1$
2. Assume $\prod_{i=0}^{2t} F_i = 2^{2^{2t+1}} - 1$
3. Multiply both sides with $F_{2t+1} \cdot F_{2t+2}$:
   - $F_{2t+1} \cdot F_{2t+2} \prod_{i=0}^{2t} F_i = \prod_{i=0}^{2(t+1)} F_i$
   - $F_{2t+1} \cdot F_{2t+2} \cdot (2^{2^{2t+1}} - 1 = ... = (2^{2^{2t+1}})^4 - 1 = 2^{2^{2(t+1)+1}} - 1$

(d)

→ The Euler's totient function $\phi(n)$: Number of integers $1 \leq d \leq n$ that are coprime to n.

(e)

For the Euler's totient function we have:

$$\phi(2^t) = 2^t \left(1 - \frac{1}{2}\right) = 2^t \cdot \frac{1}{2}$$

This gives with the Brun-Titchmarsh inequality:

$$\pi(x; 2^t, 1) \leq \frac{4x}{2^t(\log x - \log 2^t)} = \frac{x}{2^t} \cdot \frac{4}{\log x - \log 2^t} \ll \frac{x}{2^t}$$

(f)

→ Theorem 2 can be applied to any base, i.e. it is still $E(x) = O(x^{5/6})$