Problem A:

Find the roots of function: $f(x) = (2^x-1)(x^2+2x-3)$

Let us suppose that x is root of the function, then

$f(x) = 0$

$\Rightarrow (2^x-1)(x^2+2x-3) = 0$

Either

$2^x = 1$

$\therefore x = 0$

Or

$x^2+2x-3=0$

$\therefore x = -3, 1$

Hence, the required roots are $0, 1, -3$

Problem B.

Show that $3.4^n + 51$ is divisible by 3 and 9. $n \in \mathbb{N}$

Here,

Using modular congruence,

$4 \equiv 1 \pmod 3$

$\Rightarrow 4^n \equiv 1^n \pmod 3$ $[\because a \equiv b \pmod n$ implies $a^n \equiv b^n \pmod n]$

$\equiv 1 \pmod 3$

$\Rightarrow 4^n \equiv 1 \pmod 3$

But,

$3 \cdot 4^n \equiv 0 \pmod 3$

Also,

$51 \equiv 0 \pmod 3$

$\therefore 3 \cdot 4^n + 51 \equiv 0 \pmod 3$ $[\because a \equiv b \pmod n$, $c \equiv d \pmod n$ implies $a+c \equiv b+d \pmod n]$

Therefore, $3 \cdot 4^n + 51$ is divisible by 3.

We know, $3 \cdot 4^n + 51 = 3(4^n + 17)$

Now,

$$4 \equiv 1 \pmod 3$$

$$\Rightarrow 4^n \equiv 1 \pmod 3$$

$$\Rightarrow 17 \equiv 2 \pmod 3$$

$$\therefore 4^n + 17 \equiv 3 \pmod 3$$

$$\equiv 0 \pmod 3$$

$\therefore 4^n + 17 = 3k$, where $k \in \mathbb{N}$

$$\therefore 3 \cdot 4^n + 51 = 3 \cdot 3k = 9k.$$

$\therefore 3 \cdot 4^n + 51$ is divisible by 9.

Problem C:

Determine numerical value of following expression without use of calculator.

$$\sqrt{\sin\left(\frac{\pi + \log_2\left(\sqrt{2^{\pi} \cdot 2^{\pi}}\right)}{2^5 - 2^4 - 2^3}\right)^3 \sqrt{\frac{2^{4-1}}{2^{3/2}}} + \log_2\left(\log_3\left(9^{15}\right)\right) + \pi^{1+(-1)^{17}} + 1 + \frac{(-1)^5 + (-1)^{27}}{(-1)^{760}}}$$

$$= \sqrt{\sin\left(\frac{\pi+\pi}{8}\right) \cdot 2^{\left(3\cdot\frac{3}{2}\right)\frac{1}{3}} + \log_2(30+1+1) + \left(\frac{-1-1}{1}\right)}$$

$$= \sqrt{\frac{1}{\sqrt{2}} \cdot \sqrt{2} + 5 - 2}$$

$$= \sqrt{4} = \pm 2$$

Problem D:

Prove for $n \in \mathbb{R}$: $|\cos(n)| \geq 1 - \sin^2(n)$

We know,

Range of $\cos(n)$ is $[-1, 1]$

$\therefore$ ~~Rang~~ $|\cos(n)| \in [0, 1]$ i.e.

$0 \leq |\cos(n)| \leq 1 \quad \forall\, n \in \mathbb{R}$

Multiplying by $|\cos(n)|$

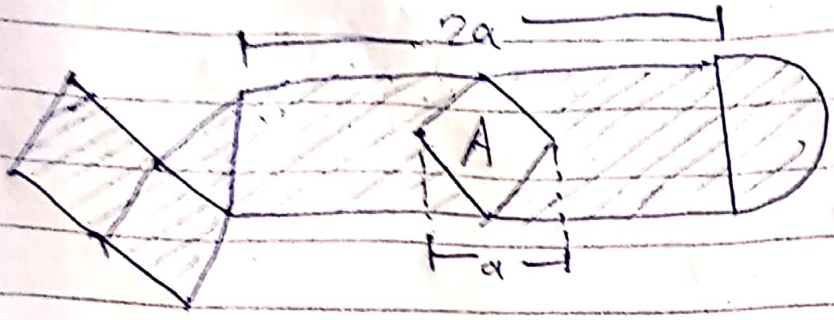
$\Rightarrow 0 \leq |\cos(n)|^2 \leq |\cos(n)|$

$\Rightarrow 0 \leq \cos^2(n) \leq |\cos(n)| \quad [\because |a|^2 = a^2]$

$\Rightarrow 1 - \sin^2(n) \leq |\cos(n)| \quad [\because \cos^2 n = 1 - \sin^2 n]$

$\therefore |\cos(n)| \geq 1 - \sin^2 n.$

Problem E.



Total area = Area of 2 squares + Area of triangles + Area of rectangle + Area of semi circle

$= 2A + A/2 + 2a^2 + \frac{\pi a^2}{8}$

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$= 2A + A/2 + 4A + \frac{\pi A}{4}$ $[\because a^2 = 2A]$

$= \frac{5A + 8A}{2} + \frac{\pi A}{4}$

$= \frac{26A + \pi A}{4}$

$\therefore$ Area of shaded region $= \frac{26A + \pi A}{4} - A$

$= \frac{22A + \pi A}{4}$