Pre-Final Round 2020

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Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend printing this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The 10 problems are separated into three categories: 4x basic problems (A; three points), 4x advanced problems (B; four points), 2x special-creativity problems (C; six points).
- You receive points for the correct solution as well as for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 40 points in total. You qualify for the final round if you reach at least 20 points (under 18 years) or 28 points (over 18 years).
- Please consider following notation that is used for the problems
  - $x, y \in \mathbb{R}$ denotes a real number, $n, k \in \mathbb{N}$ denotes a positive integer.
  - f, g, h denote functions. The domain and co-domain should follow from the context.
  - The *roots* of a function f are all x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
  - The natural logarithm of x is written as $\log(x)$.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/en/login`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no Word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sunday 15. November 2020, 23:59 UTC+0**.
- The results of the pre-final round will be announced on Monday 23. November 2020.

Good luck!

Problem A.1

Find all points (x, y) where the functions $f(x), g(x), h(x)$ have the same value:

$$f(x) = 2^{x-5} + 3, \quad g(x) = 2x - 5, \quad h(x) = \frac{8}{x} + 10$$

Problem A.2

Determine the roots of the function $f(x) = (5^{2x} - 6)^2 - (5^{2x} - 6) - 12$.

Problem A.3

Find the derivative $f'_m(x)$ of the following function with respect to x:

$$f_m(x) = \left(\sum_{n=1}^{m} n^x \cdot x^n \right)^2$$

Problem A.4

Find at least one solution to the following equation:

$$\frac{\sin(x^2 - 1)}{1 - \sin(x^2 - 1)} = \sin(x) + \sin^2(x) + \sin^3(x) + \sin^4(x) + \cdots$$

Problem B.1

Consider the following sequence of successive numbers of the 2^k-th power:

$$1,\ 2^{2^k},\ 3^{2^k},\ 4^{2^k},\ 5^{2^k},\ ...$$

Show that the difference between the numbers in this sequence is odd for all $k \in \mathbb{N}$.

Problem B.2

Prove this identity between two infinite sums (with $x \in \mathbb{R}$ and $n!$ stands for factorial):

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Problem B.3

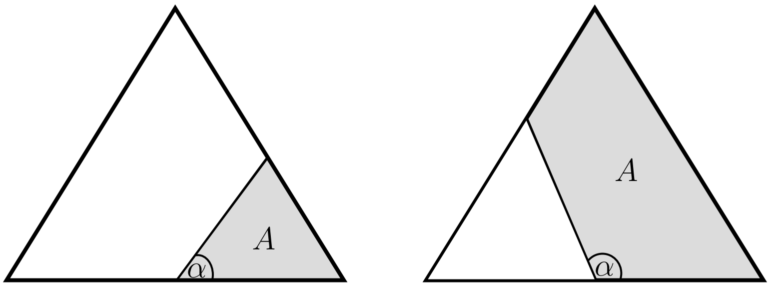
You have given a function $\lambda : \mathbb{R} \to \mathbb{R}$ with the following properties ($x \in \mathbb{R},\ n \in \mathbb{N}$):

$$\lambda(n) = 0\ , \quad \lambda(x+1) = \lambda(x)\ , \quad \lambda\left(n + \frac{1}{2}\right) = 1$$

Find two functions $p, q : \mathbb{R} \to \mathbb{R}$ with $q(x) \neq 0$ for all x such that $\lambda(x) = q(x)(p(x) + 1)$.

Problem B.4

You have given an equal sided triangle with side length a. A straight line connects the center of the bottom side to the border of the triangle with an angle of α. Derive an expression for the enclosed area $A(\alpha)$ with respect to the angle (see drawing).



Problem C.1

Let $\pi(N)$ be the number of primes less than or equal to N (example: $\pi(100) = 25$). The famous prime number theorem then states (with $\sim$ meaning *asymptotically equal*):

$$\pi(N) \sim \frac{N}{\log(N)}$$

Proving this theorem is very hard. However, we can derive a statistical form of the prime number theorem. For this, we consider *random primes* which are generated as follows:

(i) Create a list of consecutive integers from 2 to N.

(ii) Start with 2 and mark every number > 2 with a probability of $\frac{1}{2}$.

(iii) Let n be the next non-marked number. Mark every number $> n$ with a probability of $\frac{1}{n}$.

(iv) Repeat (iii) until you have reached N.

All the non-marked numbers in the list are called *random primes*.

(a) Let q_n be the probability of n being selected as a *random prime* during this algorithm. Find an expression for q_n in terms of q_{n-1}.

(b) Prove the following inequality of q_n and q_{n+1}:

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}$$

(c) Use the result from (b) to show this inequality:

$$\sum_{k=1}^{N} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1$$

(d) With this result, derive an asymptotic expression for q_n in terms of n.

(e) Let $\tilde{\pi}(N)$ be the number of *random primes* less than or equal to N. Use the result from (d) to derive an asymptotic expression for $\tilde{\pi}(N)$, i.e. the prime number theorem for *random primes*.

Problem C.2

This problem requires you to read following scientific article:

On the harmonic and hyperharmonic Fibonacci numbers.
Tuglu, N., Kızılateş, C. & Kesim, S. Adv Differ Equ (2015).
Link: https://doi.org/10.1186/s13662-015-0635-z

Use the content of the article to work on the problems (a-f) below. All problems marked with * are bonus problems (g-i) that can give you extra points. However, it is not possible to get more than 40 points in total.

(a) What are the values of H_n, F_n and $\mathbb{F}_n$ for $n = 1, 2, 3$?

(b) Determine the hyperharmonic number $H_3^{(10)}$ (Tip: use Equation 4) and $F_2^{(3)}$.

(c) Use the definition of $x^{\underline{m}}$ to simplify the following fraction: $\dfrac{x^{\underline{m+1}} - x^{\underline{m}}}{x^{\underline{m}} + x^{\underline{m+1}}}$

(d) Present the proof of Theorem 1 step-by-step by applying Equation 6.

(e) Show that $\mathbb{F}_n^{(r)} - \mathbb{F}_{n-2}^{(r)} = \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)}$.

(f) Determine the Euclidean norm of the circulant matrix $\mathrm{Circ}(1, 1, 0, 0)$.

(g*) Show that for $u(k) = \mathbb{F}_k^2$ we get $\Delta u(k) = \frac{1}{F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$.

(h*) Use the theorems from the article to prove the following identity:

$$\sum_{k=1}^{n-1} k^{\underline{m}}(\mathbb{F}_k)^2 = \frac{n^{\underline{m+1}}}{m+1}\mathbb{F}_n^2 - \sum_{k=0}^{n-1} \frac{(k+1)^{\underline{m+1}}}{(m+1)F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$$

(i*) Use Equation 1 and Theorem 5 to show the following:

$$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right)$$