International Youth Math Challenge

Qualification Round 2020 - Solution

Problem A

Considering the factors separately:

1. $2^x - 1 = 0 \quad \Rightarrow \quad x = 0$
2. $x^2 + 2x - 3 = 0 \quad \Rightarrow \quad x = 1,\ x = -3$

Solution: $x \in \{-3, 0, 1\}$

Problem B

As $3|9$, it is enough to show that $9|(3 \cdot 4^n + 51)$ by induction:

1. $3 \cdot 4 + 51 = 63$ (divisible by 9)
2. $(3 \cdot 4^{n+1} + 51) - (3 \cdot 4^n + 51) = 3 \cdot 3 \cdot 4^n = 9 \cdot 4^n$ (divisible by 9)

Problem C

Using basic calculation rules gives the following:

$$\sqrt{\sin\left(\frac{\pi + \pi}{8}\right) \cdot \frac{2}{\sqrt{2}} + \log_2(15 \cdot 2 + 2) - 2} = \sqrt{1 + 5 - 2} = 2$$

Problem D

From $\cos(x) \leq 1$ and the trigonometric Pythagoras $1 = \sin^2(x) + \cos^2(x)$ it follows:

$$1 - \sin^2(x) = \cos^2(x) \leq |\cos(x)|^2 \leq |\cos(x)|$$

Problem E

by using A

The white square side length is $s = \sqrt{A}$. It follows the height h of the rectangle: $s^2 + s^2 = h^2 \Rightarrow h = \sqrt{2}s = \sqrt{2A}$. Thus, the area of

- the rectangle area is $A_1 = 2h^2 - A$,
- the semicircle $A_2 = \pi(h/2)^2/2$,
- the triangle $A_3 = h^2/4$.

This gives the total area:

$$\begin{aligned}
A_1 + A_2 + A_3 + 2A &= 2h^2 - A + \pi(h/2)^2/2 + h^2/4 + 2A \\
&= 4A - A + \pi A/4 + A/2 + 2A \\
&= A \cdot \left(5 + \frac{\pi + 2}{4}\right) \\
&= A \cdot \left(\frac{\pi + 22}{4}\right)
\end{aligned}$$

by using a

The area of the white square $A = 2 \cdot (a^2/4) = a^2/2$. Thus, the area of

- the rectangle area is $A_1 = 2a^2 - a^2/2$,
- the semicircle $A_2 = \pi(a/2)^2/2$,
- the triangle $A_3 = a^2/4$.

This gives the total area:

$$\begin{aligned}
A_1 + A_2 + A_3 + 2A &= 2a^2 - a^2/2 + \pi(a/2)^2/2 + a^2/4 + a^2 \\
&= a^2 \cdot \left(4 + \frac{\pi + 2}{8}\right) \\
&= a^2 \cdot \left(\frac{\pi + 34}{8}\right)
\end{aligned}$$