IYMC Qualification Round

Viet Nam – Bui Phuong Tra – 17/05/2003 (Under 18)

Problem A

We have:

$\left(\sqrt{x^3}-\sqrt{x}\right)^2 \geq 0 \ (x \geq 0)$

$\leftrightarrow x^3 + x \geq 2x^2$

$\leftrightarrow -x^3 \leq -2x^2 + x$

$\leftrightarrow x + x^2 - x^3 \leq x + x^2 - 2x^2 + x = -x^2 + 2x = -(x-1)^2 + 1 \leq 1$

(because $-(x-1)^2 \leq 0 \ for \ all \ x \in R$)

$\leftrightarrow f(x) \leq 1$, " = " $when \ x = 1$

The maximum value of the function $f(x)$ is 1, which occurs when $x = 1$

Problem B

$1n^3 + 2n + 3n^2$

$= n(n^2 + 3n + 2)$

$= n(n^2 + n + 2n + 2)$

$= n[n(n+1) + 2(n+1)]$

$= n(n+1)(n+2)$

We have: $A = \ n(n+1)(n+2)$

To prove that A is divisible by 2:

* If n is an odd positive integer: $(n+1)$ will be an even positive integer

→ A is divisible by 2.

* If n is an even positive integer: n and (n+2) will be even positive integers

→ A is divisible by 2.

To prove that A is divisible by 3:

*If positive integer n is divisible by 3 → A is divisible by 3.

*When positive integer n is divided by 3, the remainder is 1 → (n+2) is divisible by 3 → A is divisible by 3.

*When positive integer n is divided by 3, the remainder is 2 → (n+1) is divisible by 3 → A is divisible by 3.

In conclusion: A always includes at least one number divisible by 2 and one number divisible by 3 → A is divisible by 2 and 3.

Or we can say that: Because n, n+1, n+2 are three consecutive positive numbers, and there are always one number divisible by 3 and at least one number divisible by 2 among three positive consecutive number (Pigeonhole Principle – Dirichlet). Therefore, A is divisible by 2 and 3.

Problem C

$$\sin\left(x + \frac{\pi^3 + 2\sqrt{\pi^6}}{\pi^2 + \pi^2} + \pi^{\pi^0}\right) = \cos\left(x + \frac{(-1)^{16}}{2} - \frac{\log_2(\sqrt{8})}{3}\right) \ (1)$$

We have:

$$\pi^0 = 1 \rightarrow \pi^{\pi^0} = \pi^1$$

$$\frac{(-1)^{16}}{2} = \frac{1}{2}$$

$$\log_2(\sqrt{8}) = \frac{1}{2}\log_2 8 = \frac{1}{2}\log_2 2^3 = \frac{3}{2} \rightarrow \log_2(\sqrt{8}) = \frac{3}{2}$$

Therefore:

$$(1) \leftrightarrow \ \sin\left(x + \frac{\pi^3 + 2\pi^{\frac{6}{2}}}{2\pi^2} + \pi^1\right) = \cos\left(x + \frac{1}{2} - \frac{3}{2} \times \frac{1}{3}\right)$$

$$\leftrightarrow \sin\left(x + \frac{3\pi^3}{2\pi^2} + \pi\right) = \cos(x)$$

$$\leftrightarrow \sin\left(x + \frac{3}{2}\pi + \pi\right) = \cos(x)$$

$$\leftrightarrow \sin\left(x + \frac{5}{2}\pi\right) = \sin\left(\frac{\pi}{2} - x\right)$$

$\leftrightarrow \sin\left(x+\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}-x\right)\ (because\ \ \sin(x) = \sin(x+2\pi))$

$\rightarrow \begin{cases} x+\frac{\pi}{2} = \frac{\pi}{2} - x + k2\pi\ (k\ is\ an\ integer) \\ x+\frac{\pi}{2} = \pi - \left(\frac{\pi}{2}-x\right) + k2\pi \end{cases}$

$\rightarrow \begin{cases} 2x = k2\pi \\ x = x + k2\pi\ (true\ for\ \forall x) \end{cases}$

$\rightarrow The\ equation\ is\ true\ for\ all\ x \in R$

In conclusion, 2 possible values of x can be π (rad) and 2π (rad).

Problem D

$\alpha + \beta + \gamma = 1\ (1)$

$\beta + \gamma + \beta = 1\ (2)$

$\gamma + \beta + \gamma = 1\ (3)$

By adding (2) to (3) we have:

$3(\beta + \gamma) = 2$

$\rightarrow \beta + \gamma = \frac{2}{3}\ (4)$

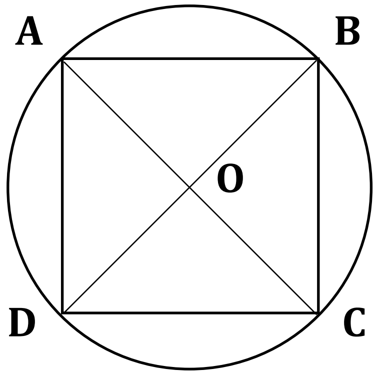
From (4) and (1), we have:

$\alpha + \frac{2}{3} = 1$

$\rightarrow \alpha = 1 - \frac{2}{3} = \frac{1}{3}$

In conclusion, $\alpha = \frac{1}{3}$.

Problem E



R is the radius of the circle.

Square ABCD is inside the circle.

We have AC = BD = 2R = 2OA = 2OB = 2OC = 2OD

The surface area of the circle is:

$$\pi R^2 = 1\ (m^2) \rightarrow R^2 = \frac{1}{\pi}(m^2) \rightarrow R = OA = OB = OC = OD = \sqrt{\frac{1}{\pi}}\,(m)$$

The square is divided into four equal parts.

The surface area of each small square part: $\frac{R^2}{2}\left(= \frac{OA.OD}{2}\right)(m^2)$

The surface area of the square ABCD is: $4 \times \frac{R^2}{2} = 2R^2 = \frac{2}{\pi}(m^2)$