I Y M C

Pre-Final Round 2019

Ziqian Xia

DATE OF RELEASE: 28. OCTOBER 2019

Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend printing this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The 10 problems are separated into three categories: 4x basic problems (A; three points), 4x advanced problems (B; four points), 2x special-creativity problems (C; six points).
- You receive points for the correct solution as well as for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 40 points in total. You qualify for the final round if you reach at least 20 points (under 18 years) or 28 points (over 18 years).
- Please consider following notation that is used for the problems
  - $x, y \in \mathbb{R}$ denotes a real number, $n \in \mathbb{N}$ denotes a positive integer.
  - f, g, h denote functions. The domain and co-domain should follow from the context.
  - The "roots" of a function f are those x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/en/login`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no Word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sunday 3. November 2019, 23:59 UTC+0**.
- The results of the pre-final round will be announced on Monday 11. November 2019.

Good luck!

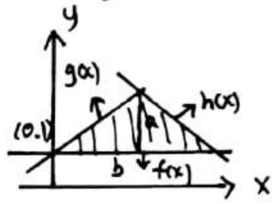
Problem A.1

Find the area enclosed be these three functions: $f(x) = 1$, $g(x) = x + 1$, $h(x) = 9 - x$

---

① Draft these three function in the Cartesian coordinates. As figure I

② The shadow area is $\boxed{S = \frac{1}{2}ab = \frac{1}{2}\times 4\times 8 = 16}$



Problem A.2

Find the roots of the function $f(x) = 3^x \cdot (\log_2(x) - 3)^5 \cdot e^{x^2-3x}$.

---

① Because the equation $3^x \neq 0$ and $e^{x^2-3x} \neq 0$

So by analyzing the factor, the root is in the $\log_2(x) - 3 = 0$

② Solve for x, $\log_2(x) = 3 \Rightarrow \boxed{x = 8}$

Problem A.3

Find the derivative $f'(x)$ of the function $f(x) = x^{\sin(x)}$.

① Compound function derivative rules:

$$f'(x) = \frac{d e^{\sin(x)\ln^x}}{dx} = \frac{e^{\sin(x)\ln^{(x)}} d \sin(x)\ln^{(x)}}{dx}$$

$$\boxed{= x^{\sin(x)} [\cos(x)\ln^{(x)} + \frac{1}{x}\sin^{(x)}]}$$

Problem A.4

Find the value of this expression for $n \to \infty$:

$$\left(\sqrt{1-\frac{1}{n}}\right)^n \cdot \sqrt{\left(1-\frac{1}{n}\right)^n}$$

Hint: You may use that $e^x = (1 + \frac{x}{n})^n$ for $n \to \infty$.

① Simplify:

$$\left(\sqrt{1-\frac{1}{n}}\right)^n \cdot \sqrt{\left(1-\frac{1}{n}\right)^n} = \left(1-\frac{1}{n}\right)^n = \frac{1}{(1+\frac{1}{n-1})} \times \frac{1}{(1+\frac{1}{n-1})^{n-1}}$$

②

Hint: $e^x = (1+\frac{x}{n})^n \quad (n \to \infty)$

$1 + \frac{1}{n} = 1 \quad (n \to \infty)$

③

$$\lim_{n \to \infty} \left(\sqrt{1-\frac{1}{n}}\right)^n \cdot \sqrt{\left(1-\frac{1}{n}\right)^n} = \boxed{\frac{1}{e}}$$

Problem B.1

Find all positive integers n such that $n^4 - 1$ is divisible by 5.

① factoring the equation as $n^4-1 = (n+1)(n-1)(n^2+1)$

② These three factor can be divided by 5 only when :

(1) $n = 5k+1$, $n = 5k+1$ $(k=1,2,3\cdots)$ for $n+1$ and n^2+1

(2) $n = 2+10m$
$n = 3+10m$ $(m = 0,1,2,3\cdots)$ for $n-1$
$n = 7+10m$
$n = 8+10m$

So all the positive integers satisfy (1), (2) is suitable.

Problem B.2

Prove the following inequality between the harmonic, geometric, and arithmetic mean with $x, y \geq 0$:

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

① Scenario I

when both x, y is >0

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

$\Downarrow$

$$x+y \geq 2\sqrt{xy}$$

$$\frac{1}{x}+\frac{1}{y} \geq \frac{2}{\sqrt{xy}}$$

$\Downarrow$

$$(\sqrt{x}-\sqrt{y})^2 \geq 0$$

$$\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{y}}\right)^2 \geq 0$$

② Scenario II

when $x=0$ or $y=0$

$$\lim_{xy\to 0^+} \sqrt{xy} = 0$$

$$\lim_{xy\to 0^+} \frac{2}{\frac{1}{x}+\frac{1}{y}} = 0$$ and ~~than~~ $0 \leq \frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy}$ $(xy\to 0^+)$

So the inequality

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$ is proved.

Problem B.3

Suppose you have to distribute the numbers $\{1, 2, 3, \dots, 2n-1, 2n\}$ over n buckets. Show that there will always be at least one bucket with its sum of numbers to be $\geq 2n+1$.

Solving by Contradiction: Assuming the buckets of each numbers is alway less than $2n+1$ and denoted as e_i $(i = 1, 2, 3 \cdots)$

$$\sum_{i=1}^{n} e_i < n(2n+1) \qquad \sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2}$$

$$\Downarrow \qquad \not\nearrow$$

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{2n} i$$

$\Rightarrow$ $n(2n+1) < n(2n+1)$ $\rightarrow$ Contradict to the Assuming

$\swarrow$

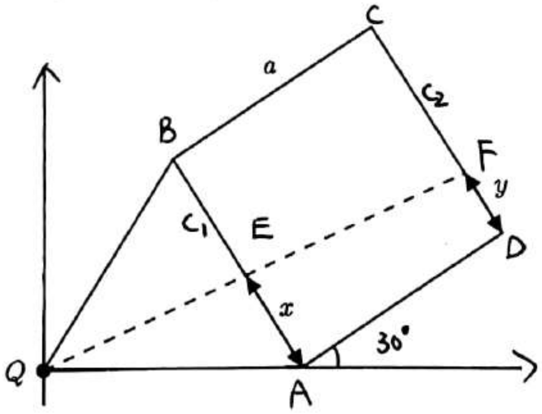
The original consideration is right

$\Downarrow$

proved.

Problem B.4

Consider an equal-sided triangle connected to a square with side a (see drawing). A straight line from Q intersects the square at x and y. You have given x, find an equation for the intersection at $y(x)$.



Choose Q As origin. estabilish Coordinate system. Denote A, B, C, D, E, F As shown in figure.

Line C_1 : $y = -\sqrt{3}x + \sqrt{3}a$ Line C_2 : $y = -\sqrt{3}x + a(2+\sqrt{3})$

$A(a, 0)$ $B(\frac{a}{2}, \frac{\sqrt{3}}{2}a)$ $C((\frac{1}{2}+\frac{\sqrt{3}}{2})a, (\frac{1}{2}+\frac{\sqrt{3}}{2})a)$. $D((1+\frac{\sqrt{3}}{2})a, \frac{a}{2})$

$E(\frac{\sqrt{3}a}{m+\sqrt{3}}, \frac{m\sqrt{3}a}{m+\sqrt{3}})$ $F(\frac{(2+\sqrt{3})m}{m+\sqrt{3}}, \frac{(2+\sqrt{3})am}{m+\sqrt{3}})$ $\Rightarrow$ $y = \frac{\sqrt{3}}{2}((2+\sqrt{3})x - a)$

$$\boxed{y(x) = \frac{\sqrt{3}}{3}[(2+\sqrt{3})x - a]} \qquad x \in [(2-\sqrt{3})a, (\sqrt{3}-1)a]$$

Problem C.1

The sum of divisor function $\sigma(n)$ returns the sum of all divisors d of the number n:

$$\sigma(n) = \sum_{d|n} d$$

We denote N_k any number that fulfils the following condition:

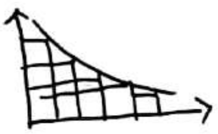
$$\sigma(N_k) \geq k \cdot N_k$$

Find examples for N_3, N_4, N_5 and prove that they fulfil this condition.

---

① $\sigma(n!) = \sum_{d|n!} d = n!(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}) + 1+2+3\cdots$

$> n!(1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{n})$

②  function $\frac{1}{x}$: $S_i > \int_i^{i+1} \frac{1}{x}dx$

$\sigma(n!) \geq kn$

So $1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{n} > \int_1^{n+1} \frac{1}{x}dx$

while $\int_1^{n+1} \frac{1}{x}dx = \ln^{(n+1)}$

$\ln^{(n+1)} \geq k \Rightarrow \sigma(n!) \geq kn! \Rightarrow n \geq e^k - 1$

③ INT (x)

Let $n = [e^k - 1] + 1$

$$\boxed{\begin{array}{l} N_3 = 20! \\ N_4 = 54! \\ N_5 = 148! \end{array}}$$

Problem C.2

This problem requires you to read following recently published scientific article:

Encoding and Visualization in the Collatz Conjecture.
George M. Georgiou, arXiv:1811.00384, (2019).
Link: `https://arxiv.org/pdf/1811.00384.pdf`

Please answer following questions related to the article:

(a) Explain the *Collatz conjecture* in your own words. Have we proven this conjecture?

(I) Any positive integer (n) can be iterated into Collatz and return 1 finally.

(II) It has not been proved yet.

(b) What is the $C(n)$ cycle and the $T(n)$ cycle of the number $n = 48$?

(I) $n=48$ $C(48) \to C(24) \to C(12) \cdots C(3) \to C(10) \to C(5) \to C(16) \cdots C(1)$

$C(n)$ cycle is $C(1) \to C(4) \to C(2) \to C(1)$

$T(n)$ cycle is $T(1) \to T(2) \to T(1)$

(c) Explain the meaning of $\sigma_\infty(n)$ and calculate $\sigma_\infty(104)$.

(I) $\sigma_\infty(n)$ means the circulation it takes to get a interger become 1.

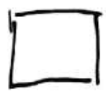
$\boxed{\sigma_\infty(104) = 10}$

(d) Find the binary encoding of $n = 32, 53, 80$ and explain why they all start with "111".

(I) 111, 011, 010, 111, 111, 011, 011, 101, 001, 010, 011, 010, 001, 101, 01011

(II) Rolling back to cycle: $1 \to 2 \to 4 \to 8$. So the ~~First~~ last three number is 842.
and the First three is "111"

(e) Make a drawing of the Collatz curve of $n = 2^{10} = 1024$.



(f) What is more common according to the data: r-curves with finite girth or acyclic r-curves?

R-curves with finite girth.