Solutions to IYMC Qualifying Round 2018

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September 29, 2018

Problem A. Find the roots of $f(x) = (e^x - e^\pi)(e^x - \pi)$ where e denotes Euler's number.

Solution. The roots of the function f could be found by equating it to 0.

$$
\begin{array}{rcl}
(e^x - e^\pi)(e^x - \pi) & = & 0 \\
(e^x - e^\pi) = 0 & \| & (e^x - \pi) = 0 \\
e^x = e^\pi & \| & e^x = \pi \\
x = \pi & \| & x = \ln \pi
\end{array}
$$

So the roots of f are $x = \pi$ and $x = \ln \pi$ ■

Problem B. Show that $n^4 - n^3 + n^2 - n$ is divisible by 2 for all positive integers n.

Proof. To show that the expression is divisible by 2 (that is even), it must be proven that the expression is even for $n = 1$ then to both even and odd values of n. An expression is even if it can be expressed as $2m$ where $m \in \mathbb{N}$.

$$\begin{aligned} n^4 - n^3 + n^2 - n &= n^4 + n^2 - n^3 - n \\ &= n^2\left(n^2+1\right) - n\left(n^2+1\right) \\ &= \left(n^2+1\right)\left(n^2-n\right) \\ &= n\left(n^2+1\right)(n-1) \end{aligned}$$

Case 1: For $n = 1$.

$$\begin{aligned} n\left(n^2+1\right)(n-1) &= 1(1^2+1)(1-1) \\ &= 1(2)(0) \\ &= 0 \end{aligned}$$

Zero is divisible by 2.

Case 2: For $n = 2k$ (for all $k \in \mathbb{N}$).

$$\begin{aligned} n\left(n^2+1\right)(n-1) &= 2k\left((2k)^2+1\right)(2k-1) \\ &= 2k\left(4k^2+1\right)(2k-1) \\ &= 2\left[k\left(4k^2+1\right)(2k-1)\right] \end{aligned}$$

Since $k \in \mathbb{N}$, $(4k^2+1) \in \mathbb{N}$. Also, the least integral solution for $(2k-1)$ to be positive is 1, so $(2k-1) \in \mathbb{N}\,\forall\, k \in \mathbb{N}$. Thus $k\,(4k^2+1)\,(2k-1) \in \mathbb{N}$. Therefore, for $n \in 2k$ for all $k \in \mathbb{N}$, $n\,(n^2+1)\,(n-1)$ could be expressed as $2m$ where $m = k\,(4k^2+1)\,(2k-1)$.

Case 3: For $n = 2k+1$ (for all $k \in \mathbb{N}$).

$$\begin{aligned} n\left(n^2+1\right)(n-1) &= (2k+1)\left((2k+1)^2+1\right)((2k+1)-1) \\ &= (2k+1)\left((2k+1)^2+1\right)(2k) \\ &= 2\left[k\,(2k+1)\left((2k+1)^2+1\right)\right] \end{aligned}$$

Since $k \in \mathbb{N}$, $(2k+1) \in \mathbb{N}$, so $\left((2k+1)^2+1\right) \in \mathbb{N}$ as well. Thus $k\,(2k+1)\left((2k+1)^2+1\right) \in \mathbb{N}$. Therefore, for $n \in 2k+1$ for all $k \in \mathbb{N}$, $n\,(n^2+1)\,(n-1)$ could be expressed as $2m$ where $m = k\,(2k+1)\left((2k+1)^2+1\right)$.

Since $n\,(n^2+1)\,(n-1)$ is even for $n = 1$, even $(n = 2k)$, and odd $(n = 2k+1)$ positive integers, $n^4 - n^3 + n^2 - n$ is also even; therefore, divisible by 2 for all positive integer n

□

Problem C. You have given a sphere with a volume of π^3. What is the radius of this sphere? Explain whether or not it is possible to build such a sphere in reality?

Solution. Solving for the radius r,

$$\begin{aligned}
\pi^3 &= \frac{4\pi}{3}r^3 \\
3\pi^3 &= 4\pi r^3 \\
r^3 &= \frac{3\pi^3}{4\pi} \\
r &= \sqrt[3]{\frac{3\pi^2}{4}}
\end{aligned}$$

Since the radius of the sphere in question contains the number π, it will be impossible to create a physical object that has dimensions with incredible precision to contain the length π, let alone its cube root. Also, since π is a transcendental number, it is impossible to be constructed using Euclidean tools; thus impossible to be constructed in the real world. ■

Problem D. Find the numerical value of the following expression without the use of a calculator.

$$\log_2\left(2^2 + 5 \cdot 2^2 \cdot 3\right) \cdot \left(2\log_3 2 + \log_3\left(7 - \frac{1}{4}\right)\right) + \frac{\left(\log_2 128 - 2\right)^3}{3+2} + (-1)^{32+\pi^0}$$

Solution. Using laws of logarithms, the expression could be simplified.

$$\log_2\left(2^2 + 5 \cdot 2^2 \cdot 3\right) \cdot \left(2\log_3 2 + \log_3\left(7 - \frac{1}{4}\right)\right) + \frac{\left(\log_2 128 - 2\right)^3}{3+2} + (-1)^{32+\pi^0}$$

$$\log_2(4+60) \cdot \left(\log_3 4 + \log_3\left(\frac{27}{4}\right)\right) + \frac{\left(\log_2 2^7 - 2\right)^3}{5} + (-1)^{32+1}$$

$$\log_2(64) \cdot \left(\log_3 4 + (\log_3 27 - \log_3 4)\right) + \frac{(7-2)^3}{5} + (-1)^{33}$$

$$\log_2 2^6 \cdot \left(\log_3 3^3\right) + \frac{(5)^3}{5} + (-1)$$

$$6 \cdot 3 + 5^2 - 1$$

$$18 + 25 - 1$$

$$42$$

■

Problem E. A square has a side length a. A line intersects the square at a height of x and y. Find an expression for the surface area $A(x, y)$ below the line.

Solution. The region beneath the line can be generally described as a trapezoid with height a and bases of lengths x and y. So the area (1) given by the formula

$$A(x, y) = \frac{a(x + y)}{2} \tag{1}$$

This equation is still valid since a is just a constant. Considering special cases, this equation for the area still holds true.

Case 1: When $x = y$, the region enclosed beneath the line is a rectangle. So the area of the rectangle A is given by $A = bh = ax = ay$. Plugging $x = y$ to (1) gives the following

$$\begin{aligned} A(x, x) &= \frac{a(x + x)}{2} \\ &= \frac{a(2x)}{2} \\ &= ax \end{aligned}$$

which is equal to the area using the formula for area of a rectangle.

Case 2: When $x = y = a$, the region enclosed by the line is the entire square, so it is expected that the area is a^2. Plugging in the value for this case gives

$$\begin{aligned} A(a, a) &= \frac{a(a + a)}{2} \\ &= \frac{a(2a)}{2} \\ &= a^2 \end{aligned}$$

which equals the desired area of a square.

Case 3: When $x = 0$ or $y = 0$, the region formed is a triangle, so the area of the region is given by $A = \frac{bh}{2} = \frac{ax}{2}$ or $\frac{ay}{2}$. Without loss of generality, let $y = 0$, so the area as given by (1) with this value is

$$\begin{aligned} A(x, 0) &= \frac{a(x + 0)}{2} \\ &= \frac{ax}{2} \end{aligned}$$

which equals the area given by a triangle.

Case 4: When $x = y = 0$, the expected area is 0. Substituting to (1) gives

$$
\begin{aligned}
A(0,0) &= \frac{a(0+0)}{2} \\
&= 0
\end{aligned}
$$

which corresponds with the expected area.

These cases verify that (1) will work as the area of the region below the line. ■