Problem A.1

Find the roots of the function $f(x) = 2^3x^3 + 2^2x^2 + 2x + 1$.

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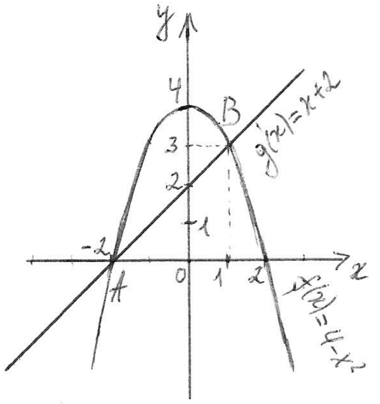
1. We need to transform the function $f(x)$ into a convenient form:
$f(x) = 2^3x^3 + 2^2x^2 + 2x + 1 = 2^2x^2(2x+1) + 2x + 1 = (2x+1)(2^2x^2+1) = (2x+1)(4x^2+1)$
2. $f(x) = 0 \Rightarrow (2x+1)(4x^2+1) = 0$
So, clearly either $(2x+1) = 0$ or $(4x^2+1) = 0$ (or both). But we know that $x^2 \geq 0 \Rightarrow 4x^2 \geq 0 \Rightarrow 4x^2+1 > 0$. So $(4x^2+1)$ can not be equal to zero for $\forall x$.
3. $2x+1=0$
$x = -\frac{1}{2}$ – the root of the given function $f(x)$.

We can check it out: $f(-\frac{1}{2}) = 2^3\cdot(-\frac{1}{2})^3 + 2^2(-\frac{1}{2})^2 + 2\cdot(-\frac{1}{2}) + 1 = -1+1-1+1 = 0$

Answer: $x = -\frac{1}{2}$

Problem A.2

Draw the functions $f(x) = 4 - x^2$ and $g(x) = x + 2$ and find the points of intersection (x, y).



1. As we see the functions $f(x) = 4 - x^2$ and $g(x) = x + 2$ have two points of intersection: point A and point B.
2. Coordinates x_A and x_B of the points of intersection can be found from:
$4 - x^2 = x + 2$
$x^2 + x - 2 = 0$
$x_1 = 1,\ x_2 = -2$
From the drawing: $x_A = x_2 = -2$; $x_B = x_1 = 1$.
3. Then we can find the coordinates y_A and y_B:
$y_A = f(x_A) = g(x_A) = x_A + 2 = -2 + 2 = 0$
$y_B = f(x_B) = g(x_B) = x_B + 2 = 1 + 2 = 3$
$A(-2; 0)$, $B(1; 3)$

Answer: $(-2; 0)$, $(1; 3)$ – points of intersection

Problem A.3

Find the derivative $f'(x)$ of the function $f(x) = 2^x \cdot x^2$.

$f'(x) = (2^x \cdot x^2)' = [(u \cdot v)' = u'v + uv'] = (2^x)' \cdot x^2 + 2^x \cdot (x^2)' =$

$= 2^x \ln 2 \cdot x^2 + 2^x \cdot 2x = 2^x \cdot x(x \ln 2 + 2)$

Answer: $2^x x(x \ln 2 + 2)$

Problem A.4

Determine all x that solve the equation $x^{2x} + 27^2 = 54x^x$.

$x^{2x} + 27^2 = 54x^x$

$(x^x)^2 - 54x^x + 27^2 = 0$

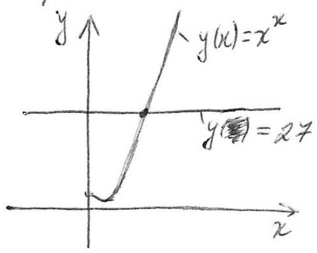
$(x^x - 27)^2 = 0$

$x^x - 27 = 0$

$x^x = 27$

$x^x = 3^3 \Rightarrow x = 3$

Lets prove that there are no other roots

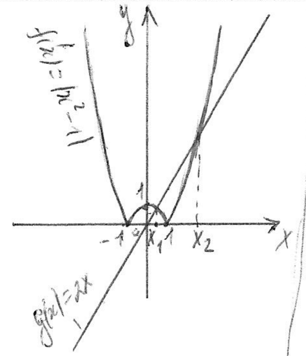


We know what the function $y(x) = x^x$ looks like. And clearly that there is only one point of intersection between $y(x) = x^x$ and $y(x) = 27$ with the coordinates $(3;3)$.

Answer: $x = 3$

Problem A.5

Find all x such that $|x^2 - 1| < 2x$.



1. $(x^2-1) \geq 0 \Rightarrow x^2-1 < 2x \Rightarrow x^2-2x-1<0$

$-1 \quad 1 \quad x$

$\mathcal{D} = (-2)^2 - 4\cdot1\cdot(-1) = 4+4 = 8, \ \sqrt{\mathcal{D}} = \sqrt{8} = 2\sqrt{2}$

$x_{1,2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \Rightarrow (x-x_1)(x-x_2) < 0$

$(1-\sqrt{2}) \quad 0 \quad (1+\sqrt{2}) \quad x$

Only $x \in [1; 1+\sqrt{2})$ satisfies $x^2-1 \geq 0$ and $x^2-1<2x$

2. $x^2-1 < 0 \Rightarrow -(x^2-1) < 2x \Rightarrow x^2+2x-1 > 0$

$1 \quad 1$

$\mathcal{D} = 2^2 - 4\cdot(-1) = 8, \ \sqrt{\mathcal{D}} = \sqrt{8} = 2\sqrt{2}$

$x_{3,4} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \Rightarrow (x-x_3)(x-x_4) > 0$

$-1-\sqrt{2} \quad 0 \ (-1+\sqrt{2})$

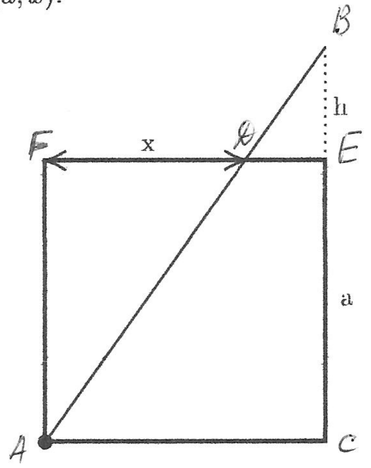
Only $x \in (-1+\sqrt{2}; 1]$ satisfies $x^2-1<0$ and $-(x^2-1)<2x$

Answer: $x \in (-1+\sqrt{2}; 1+\sqrt{2})$

Combine all: $x_3 \quad 1 \quad x_1 \quad 0 \quad x_4 \quad 1 \quad x_2 \quad x$

Problem A.6

You have given a square with side a and an intersecting straight line in a distance of x as seen below. Find an equation for the height $h(a,x)$.



1. $\triangle ABC, \angle ACB = 90^\circ$; $\triangle DBE, \angle DEB = 90^\circ$

$\angle BAC = \angle BDE$

So $\triangle ABC \sim \triangle DBE$ and $\frac{AC}{DE} = \frac{BC}{BE} = \frac{AB}{DB}$

2. $AFEC$ – a square $\Rightarrow AF = FE = EC = AC = a$

$DE = FE - x = a - x$

3. $\frac{AC}{DE} = \frac{BC}{BE} \Rightarrow \frac{a}{a-x} = \frac{h+a}{h}$

$ah = (a-x)(h+a) \Rightarrow ah = ah - xh - ax + a^2 \Rightarrow a^2 - xh - ax = 0$

$h = \frac{a^2 - ax}{x} = \frac{a(a-x)}{x} = \frac{a}{x}(a-x)$

Answer: $h(a,x) = \frac{a}{x}(a-x)$

Problem B.1

Show that $2^{3n} - 1$ is divisible by 7 for all positive integers n.

$2^{3n}-1=(2^3)^n-1=(2^3-1)((2^3)^{n-1}+(2^3)^{n-2}\cdot 1+(2^3)^{n-3}\cdot 1^2+\dots+(2^3)^2\cdot 1^{n-3}+2^3\cdot 1^{n-2}+1^{n-1})=$

$=7\cdot(8^{n-1}+8^{n-2}+8^{n-3}+\dots+8^2+8+1).$

Clearly that that expression is divisible by 7 for all positive integers n

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Problem B.2

Determine the biggest value of the function $f(x) = e^{-x}\sin(x)$ for $x \geq 0$.

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1. The biggest value of the function is either an extremum of the function, or $f(0)$, or $f(\infty)$. Well-known that $f'(x)=0$ for the points of extremum (max or min).

$f'(x)=(e^{-x})'\sin x+e^{-x}\cdot(\sin x)'=-e^{-x}\sin x+e^{-x}\cos x$

$-e^{-x}(\sin x-\cos x)=0 \Rightarrow e^{-x}>0$ for all $x \Rightarrow \sin x-\cos x=0$

If $\cos x=0$ then $\sin x=\pm 1$ and $\sin x-\cos x=\pm 1-0\neq 0$. So we can divide $(\sin x-\cos x)$ by $\cos x$:

$\operatorname{tg} x=1 \Rightarrow x=\frac{\pi}{4}+\pi n,\ n\in\mathbb{Z}$ – the points of extremum

2. $\sin(\frac{\pi}{4}+2\pi n)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$\sin(\frac{3\pi}{4}+2\pi n)=\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$ $\Rightarrow$ we must consider only $x=\frac{\pi}{4}+2\pi n,\ n\in\mathbb{Z}$ becouse $f(\frac{3\pi}{4}+2\pi n)<0$ and $f(\frac{\pi}{4}+2\pi n)>0$.

3. Clearly that $\sin(\frac{\pi}{4})=\sin(\frac{\pi}{4}+2\pi)=\dots=\frac{\sqrt{2}}{2}$

But $e^{-x}=\frac{1}{e^{x}}$ and the bigger x, the bigger e^{x} $\Rightarrow$ the smaller e^{-x}.

So we must determine the biggest value of e^{-x} for $x=\frac{\pi}{4}+2\pi n,\ n\in\mathbb{Z}$.

The biggest value of e^{-x} will be for the smaller value of $x=\frac{\pi}{4}+2\pi n,\ n\in\mathbb{Z}$, i.e. for $x=\frac{\pi}{4}$.

4. $f(\frac{\pi}{4})=e^{-\frac{\pi}{4}}\cdot\sin\frac{\pi}{4}=e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}$

$f(0)=e^{-0}\cdot\sin 0=0$, $f(\infty)=e^{-\infty}\sin(\infty)=\frac{1}{e^{\infty}}\cdot\sin(\infty)=0$ because $\frac{1}{e^{\infty}}=0$ and $\sin(\infty)\in[-1;1]$.

Answer: $e^{-\pi/4}\cdot\frac{\sqrt{2}}{2}$ – the biggest value of the given function

Problem B.3

Find the value of this infinite sum: $\sum_{n=0}^{\infty} \frac{2^{2n}+2^n}{2^{3n}}$.

$$\sum_{n=0}^{\infty}\frac{2^{2n}+2^n}{2^{3n}} = \sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{1}{2^{2n}}\right) = \sum_{n=0}^{\infty}\frac{1}{2^n}+\sum_{n=0}^{\infty}\frac{1}{2^{2n}} = \sum_{n=0}^{\infty}\frac{1}{2^n}+\sum_{n=0}^{\infty}\frac{1}{4^n} =$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \left(1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\dots\right)+\left(1+\left(\frac{1}{4}\right)^1+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\dots\right)$$

A common ratio of the first sum is $\frac{1}{2}<1$ and a common ratio of the second sum is $\frac{1}{4}<1$ - we have the sum of infinite geometric progression with the absolute values of the common ratios <1.

So:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{b_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{b_1}{1-q} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$\sum_{n=0}^{\infty}\frac{2^{2n}+2^n}{2^{3n}} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = 2+\frac{4}{3} = \frac{10}{3}$$

Answer: $\frac{10}{3}$

Problem B.4

Give a closed expression for the function $g(n)$ with the following behaviour:

$$g(n) = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

Clearly that the equal expression is $g(n) = n \cdot \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$.

So my first thought is that the function might contain $\sin()$, or $\cos(\dots)$ as a multiplier. It can be something like $g(n) = n\cdot\sin(\dots)$, or $g(n) = n\cdot\cos(\dots)$

Lets consider $g(n) = n\cdot\sin(w(n)) \Rightarrow \sin(w(n)) = 0$, n even and $\sin(w(n)) = 1$, n odd. - must be.

We know that $\sin x = 0$ for $x = \pi n$ and $\sin y = 1$ for $y = \frac{\pi}{2}+2\pi n$, $n\in\mathbb{Z}$

If $w(n) = \frac{\pi n}{2}$ then $\sin\frac{\pi n}{2} = \begin{cases} 0, & n \text{ even} \\ \pm 1, & n \text{ odd} \end{cases}$ - almost satisfies the given conditions

Lets try the absolute value of $\sin\frac{\pi n}{2}$:

$\left|\sin\left(\frac{n\pi}{2}\right)\right| = \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$ - we found our multiplier

So $g(n) = n\left|\sin\left(\frac{\pi n}{2}\right)\right|$ is desired function.

Similarly we can find that the function $g(n) = n\left|\left|\cos\left(\frac{\pi n}{2}\right)\right|-1\right|$ has also the asked behaviour.

Answer: $g(n) = n\left|\sin\left(\frac{\pi n}{2}\right)\right|$ or $g(n) = n\left|\left|\cos\left(\frac{\pi n}{2}\right)\right|-1\right|$

Problem B.5

Find a function $\omega(x)$ such that the function $f(x) = \sin(\omega(x))$ has the roots at $\pi, \pi^2, \pi^3, \ldots$.

$\sin(\omega(\pi)) = 0 \Rightarrow \omega(\pi) = \pi n, n \in \mathbb{Z}$ $\sin(\omega(\pi^3)) = 0 \Rightarrow \omega(\pi^3) = \pi k, k \in \mathbb{Z}$

$\sin(\omega(\pi^2)) = 0 \Rightarrow \omega(\pi^2) = \pi m, m \in \mathbb{Z}$

So we see that the function $\omega(x)$ must return $\pi l, l \in \mathbb{Z}$ for all $x = \pi, x = \pi^2, x = \pi^3, \ldots$. Such function can be a ln, lg or log.

$\omega(x) = \log_a(x) \cdot \pi \Rightarrow \omega(\pi) = \pi \log_a \pi$, $\omega(\pi^2) = \pi \log_a \pi^2 = 2 \log_a \pi$, $\omega(\pi^3) = \pi 3 \log_a \pi = 3\pi \log_a \pi$

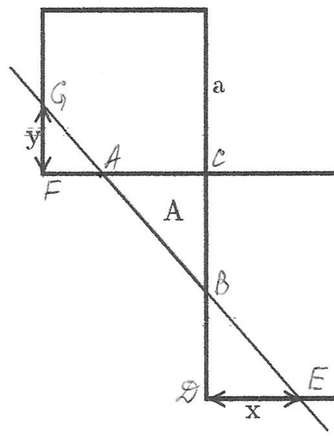
Clearly that a must be π for $\log_a \pi = 1$.

So $\omega(x) = \pi \log_\pi(x)$. Then: $f(\pi) = \sin(\pi \log_\pi \pi) = \sin \pi = 0$; $f(\pi^2) = \sin(\pi \log_\pi \pi^2) = \sin 2\pi = 0$

Answer: $\omega(x) = \pi \log_\pi(x)$.

Problem B.6

The drawing below shows two squares with side a and b. A straight line intersects the squares at y and x (see drawing). Calculate the gray area $A(a, b, x, y)$ between the squares and the line.



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1. $\triangle ACB, \angle C = 90°$; $\triangle DBE, \angle D = 90°$; $\triangle GFA, \angle F = 90°$

$\angle FAG = \angle BAC$, $\angle ABC = \angle DBE$ $=$ $\angle FAG = \angle BAC = \angle DEB$ and $\triangle ACB \sim \triangle DBE \sim \triangle GFA$.

$$\begin{cases} \dfrac{AC}{AF} = \dfrac{BC}{GF} \\ \dfrac{AC}{DE} = \dfrac{BC}{DB} \end{cases} \Rightarrow \begin{cases} \dfrac{AC}{AF} = \dfrac{BC}{y} \\ \dfrac{AC}{x} = \dfrac{BC}{DB} \end{cases} \quad \begin{matrix} BD = b - BC \\ AF = a - AC \end{matrix} \Rightarrow \begin{cases} \dfrac{AC}{a - AC} = \dfrac{BC}{y} \\ \dfrac{AC}{x} = \dfrac{BC}{b - BC} \end{cases} \Rightarrow \begin{cases} AC \cdot y = BC \cdot a - AC \cdot BC \\ AC \cdot b - AC \cdot BC = BC \cdot x \end{cases} \Rightarrow$$

$$\begin{cases} \dfrac{AC}{BC} = \dfrac{a+x}{b+y} \\ 2AC \cdot BC = BC(a-x) + AC(b-y) \end{cases} \Rightarrow \begin{cases} 2AC = a - x + (b-y) \cdot \dfrac{a+x}{b+y} \\ 2BC = b - y + (a-x) \dfrac{b+y}{a+x} \end{cases}$$

2. $A(a,b,x,y) = \frac{1}{2} AC \cdot BC = \frac{1}{8}(2AC \cdot 2BC) = \left((a-x) + (b-y)\frac{a+x}{b+y}\right)\left(b - y + (a-x)\frac{b+y}{a+x}\right) \cdot \frac{1}{8} =$

$= \frac{1}{8} \frac{ab - bx + ay - xy + ba + bx - ay - xy}{b+y} \cdot \frac{ab - ay + bx - xy + ab - bx + ay - xy}{a+x} = \frac{1}{2} \frac{(ab - xy)^2}{(b+y)(a+x)}$

Answer: $A(a,b,x,y) = \frac{1}{2} \frac{(ab - xy)^2}{(b+y)(a+x)}$