Pre-Final Round 2023

EXAMPLE SOLUTION

Problem A.1

You are given the following three functions:

$$f(x) = 3 + 3x \qquad g(x) = 2 + 2x \qquad h(x) = 5 - x$$

Find $a, b \in \mathbb{R}$ such that $h(x) < f(x)$ and $h(x) > g(x)$ for all x with $a < x < b$.

Solution: Based on the constant and slope it is $f > h > g$ for $x \in (a, b)$:

$$h(a) = f(a) \implies 5 - a = 3 + 3a \implies a = \frac{1}{2}$$

$$h(b) = g(b) \implies 5 - b = 2 + 2b \implies b = 1$$

Problem A.2

Find the derivative $f'(x)$ of the following function with respect to x:

$$f(x) = (1 + x^2)^x$$

Solution: With $f(x) = e^{x \cdot \log(1+x^2)}$ we get:

$$\begin{aligned} f'(x) &= \left(1 \cdot \log(1 + x^2) + x \cdot \frac{2x}{1 + x^2}\right) \cdot e^{x \cdot \log(1+x^2)} \\ &= \left(\log(1 + x^2) + \frac{2x^2}{1 + x^2}\right) \cdot (1 + x^2)^x \end{aligned}$$

Problem B.1

Show that this infinite sum converges and determine its value:

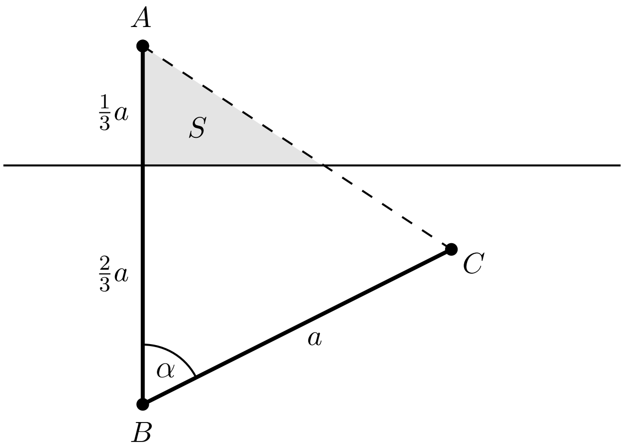
$$\sum_{n=1}^{\infty} \frac{5^n + 5^{n+1}}{5^{2n+1}}$$

Solution: Observing the geometric series gives us:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5^n + 5^{n+1}}{5^{2n+1}} &= \sum_{n=1}^{\infty} \frac{(1+5)\cdot 5^n}{5^{2n+1}} \\
&= \sum_{n=1}^{\infty} \frac{6\cdot 5^n}{5^{2n+1}} \\
&= \frac{6}{5}\cdot \sum_{n=1}^{\infty} \frac{5^n}{5^{2n}} \\
&= \frac{6}{5}\cdot \sum_{n=1}^{\infty} \frac{1}{5^n} \\
&= \frac{6}{5}\cdot \left(\sum_{n=0}^{\infty} \frac{1}{5^n} - 1\right) \\
&= \frac{6}{5}\cdot \left(\frac{1}{1-\frac{1}{5}} - 1\right) \\
&= \frac{6}{20} \\
&= \frac{3}{10}
\end{aligned}$$

Problem B.2

A vertical line $\overline{AB}$ with length a is intersected by a horizontal line at $\frac{2}{3}a$. Another line $\overline{BC}$ with length a is rotated by an angle of α and attached to point B. Find an equation for the enclosed area $S(\alpha)$ between the horizontal line and the line $\overline{AC}$ for $\arccos(2/3) \leq \alpha \leq \pi$.



Solution: The area $S(\alpha)$ of the triangle can be determined with the horizontal base $b(\alpha)$:

$$S(\alpha) = \frac{1}{2} \cdot \frac{a}{3} \cdot b(\alpha)$$

Assuming a coordinate system with origin at point A, then point C is at $(a \cdot \sin \alpha,\ a \cdot \cos \alpha - a)$, which gives for the base

$$\frac{a \cdot \cos \alpha - a}{a \cdot \sin \alpha} \cdot b = -\frac{a}{3} \quad \Longrightarrow \quad b = \frac{a}{3} \cdot \frac{\sin \alpha}{1 - \cos \alpha}$$

and thus

$$S(\alpha) = \frac{a^2}{18} \cdot \frac{\sin \alpha}{1 - \cos \alpha}$$

for $\arccos(2/3) \leq \alpha \leq \pi$ (i.e., for point C below the horizontal line).

Problem C.1

For this problem, consider the following list of seven mysterious increasing positive integers:

n	$f(n)$
1	60
2	504
3	2160
4	18144
5	77760
6	653184
7	2799360

(a) Find at least four *properties*[1] that all seven numbers have in common.

(b) Explain the underlying pattern and give a function $f(n)$ to calculate the nth number
  i. with recursion.
  ii. without recursion.

(c) What are the numerical values of the 9th and 15th number?

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[1] *Properties* can be any nontrivial characteristic related to the digits, specific type of number, certain patterns, divisibility, etc. – be creative!

Solution a: Possible answers: all even numbers/divisible by 2; all divisible by 3; all divisible by 6; end with digit 0 or 4; every second number ends with digits 60; in binary: number of leading digits equal to n; each number divides every second subsequent number; ratio $f(n)/f(n-1)$ alternates between two fractional values (4.285..., 8.4); ratio $f(n)/f(n-2)$ always 36;

Solution b: Distinct pattern in the prime factorization:

n	$f(n)$	2^x	3^x	5^x	7^x
1	60	2	1	1	0
2	504	3	2	0	1
3	2160	4	3	1	0
4	18144	5	4	0	1
5	77760	6	5	1	0
6	653184	7	6	0	1
7	2799360	8	7	1	0

(i) As we observe that $f(n)/f(n-2) = 36$, we get as a possible recursive function:

$$f(n) = 36 \cdot f(n-2)$$

(ii) Observing the increasing and alternating prime factorization, we get

$$f(n) = 2^{n+1} \cdot 3^n \cdot 5^{A(n+1)} \cdot 7^{A(n)}$$

where $A(n) = \frac{1+(-1)^n}{2}$ or $A(n) = \frac{1}{2}\left(\cos(\pi n) + 1\right)$.

Solution b: 9th: 100776960; 15th: 4701849845760

Problem C.2

This problem requires you to read following scientific article:

On Sándor's Inequality for the Riemann Zeta Function.
Alzer, H., Kwong, M. K. *Journal of Integer Sequences*, 26 (2023).
Link: https://cs.uwaterloo.ca/journals/JIS/VOL26/Alzer/alzer15.pdf

Use the content of the article to work on the problems (a-f) below:

(a) What is the numerical value of $\omega(100)$, $\zeta(2)$ and $\zeta_2(2)$?

$\longrightarrow \omega(100) = 2, \quad \zeta(2) = \pi^2/6, \quad \zeta_2(2) = 5/4$

(b) Show that $\dfrac{\zeta^3(5)}{\zeta(15)} < \sum_{n=1}^{\infty} \dfrac{6^{\omega(n)}}{n^5}$ by applying Theorem 1.

$$\longrightarrow \frac{\zeta^3(5)}{\zeta(15)} < \sum_{n=1}^{\infty} \frac{(2^3-2)^{\omega(n)}}{n^5} = \sum_{n=1}^{\infty} \frac{6^{\omega(n)}}{n^5}$$

(c) Let $\gcd(m,n) = 1$. Explain why $\omega(mn) = \omega(m) + \omega(n)$ and $F_a(mn) = F_a(m)F_a(n)$.

$\longrightarrow$ As $\gcd(m,n) = 1$, all primes in m and different from n; thus $\omega(mn) = \omega(m) + \omega(n)$.
$\longrightarrow F_a(mn) = a^{\omega(mn)} = a^{\omega(m)} \cdot a^{\omega(n)} = F_a(m)F_a(n)$

(d) Prove that $\sum_{n=1}^{\infty} \dfrac{F_a(p^n q^n)}{(p^n q^n)^s} = a^2 \dfrac{(pq)^{-s}}{1-(pq)^{-s}}$ for two distinct prime numbers p and q.

$$\longrightarrow \sum_{n=1}^{\infty} \frac{F_a(p^n q^n)}{(p^n q^n)^s} = \sum_{n=1}^{\infty} \frac{a^{\omega(p^n q^n)}}{((pq)^s)^n} = \sum_{n=1}^{\infty} \frac{a^2}{((pq)^s)^n} = a^2 \cdot \left(\tfrac{1}{1-(pq)^{-s}} - 1\right) = a^2 \frac{(pq)^{-s}}{1-(pq)^{-s}}$$

(e) In equation (11), derive explicitly why the (left/first) equality holds true.

$$\longrightarrow \sum_{n=1}^{\infty} \frac{(2^\lambda-2)^{\omega(n)}}{n^s} = \prod_{p \text{ prime}} \left(1 + \sum_{k=1}^{\infty} \frac{(2^\lambda-2)^{\omega(p^k)}}{p^{ks}}\right) = \prod_{p \text{ prime}} \left(1 + (2^\lambda-2)\frac{p^{-s}}{1-p^{-s}}\right) = RHS$$

(f) Why do (11), (12), (13) imply that equation (4) is valid?

$$\longrightarrow \sum_{n=1}^{\infty} \frac{(2^\lambda-2)^{\omega(n)}}{n^s} < \prod_{p \text{ prime}} \frac{1-p^{-s\lambda}}{(1-p^{-s})^\lambda} = \frac{\prod_{p \text{ prime}} 1-p^{-s\lambda}}{\prod_{p \text{ prime}} (1-p^{-s})^\lambda} = \frac{\zeta^\lambda(s)}{\zeta(\lambda s)} < \sum_{n=1}^{\infty} \frac{\lambda^{\omega(n)}}{n^s}$$