Pre-Final Round 2022

EXAMPLE SOLUTION

Problem A.1

Determine A, B, C such that all of the following functions intersect the point $(2, 2)$:

$$f_1(x) = Ax + 1 \qquad f_2(x) = Bx^2 + 2 \qquad f_3(x) = Cx^3 + 3$$

Solution:

$$2 = f_1(1) = 2A + 1 \implies A = 1/2$$
$$2 = f_2(1) = 4B + 2 \implies B = 0$$
$$2 = f_3(1) = 8C + 3 \implies C = -1/8$$

Problem A.2

Find all $x \in \mathbb{R}$ that are solutions to this equation: $0 = (1 - x - x^2 - ...) \cdot (2 - x - x^2 - ...)$

Solution: The RHS is divergent for $|x| \geq 1$, thus:

$$f(x) = \left(2 - \frac{1}{1-x}\right)\left(3 - \frac{1}{1-x}\right)$$

This gives us the roots of the function:

$$\frac{1}{1-x} = 2 \implies x = \frac{1}{2}$$

$$\frac{1}{1-x} = 3 \implies x = \frac{2}{3}$$

Problem A.3

Find the derivative $f'(x)$ of the following function with respect to x:

$$f(x) = \sin\left(\pi^{\sin x} + \pi^{\cos x}\right)$$

Solution:

$$\begin{aligned} f'(x) &= \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \left[\pi^{\sin x} + \pi^{\cos x}\right]' \\ &= \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \left(\cos(x)\log(\pi)\pi^{\sin x} - \sin(x)\log(\pi)\pi^{\cos x}\right) \\ &= \cos\left(\pi^{\sin x} + \pi^{\cos x}\right) \cdot \log(\pi) \cdot \left(\cos(x)\pi^{\sin x} - \sin(x)\pi^{\cos x}\right) \end{aligned}$$

Problem B.1

Let H_n define the sum of reciprocals of all integers from 1 to n:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$$

Prove the following identity:

$$H_{2n} - H_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm ... + \frac{1}{2n-1} - \frac{1}{2n}$$

Solution:

$$\begin{aligned}
H_{2n} - H_n &= \sum_{i=1}^{2n} \frac{1}{i} - \sum_{k=1}^{n} \frac{1}{k} \\
&= \sum_{k=1}^{n} \frac{1}{2k-1} + \sum_{k=1}^{n} \frac{1}{2k} - \sum_{k=1}^{n} \frac{1}{k} \\
&= \sum_{k=1}^{n} \frac{1}{2k-1} + \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{k} \right) \\
&= \sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k} \\
&= \sum_{i=1}^{2n} \frac{(-1)^{i+1}}{i} \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm ... + \frac{1}{2n-1} - \frac{1}{2n}
\end{aligned}$$

Problem B.2

It is well known that squared brackets do not simply square the individual terms:

$$(1+2)^2 \neq 1^2 + 2^2$$

$$(1+2+3)^2 \neq 1^2 + 2^2 + 3^2$$

Instead, we add a correction term ψ to make the equations hold true:

$$(1+2)^2 = 1^2 + 2^2 + \psi_2$$

$$(1+2+3)^2 = 1^2 + 2^2 + 3^2 + \psi_3$$

$$...$$

$$(1+2+3+...+n)^2 = 1^2 + 2^2 + 3^2 + ... + n^2 + \psi_n$$

Show that the correction term ψ_n has the following form and determine the values of α and β:

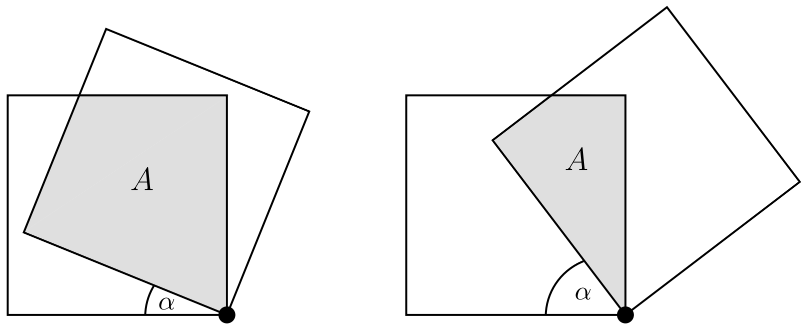
$$\psi_n = \frac{n^4 - n^2}{\alpha} + \frac{n^3 - n}{\beta}$$

Solution: We get $\alpha = 4$ and $\beta = 6$:

$$\begin{aligned}
\psi_n &= (1+2+3+...+n)^2 - 1^2 + 2^2 + 3^2 + ... + n^2 \\
&= \left(\sum_{k=1}^{n} k\right)^2 - \sum_{k=1}^{n} k^2 \\
&= \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \\
&= \frac{1}{12}\left(3n^2(n+1)^2 - 2n(n+1)(2n+1)\right) \\
&= \frac{1}{12}\left(3n^4 + 2n^3 - 3n^2 - 2n\right) \\
&= \frac{n^4 - n^2}{4} + \frac{n^3 - n}{6}
\end{aligned}$$

Problem B.3

You are given two overlaying squares with side length a. One of the squares is fixed at the bottom right corner and rotated by an angle of α (see drawing). Find an expression for the enclosed area $A(\alpha)$ between the two squares with respect to the rotation angle α.



Solution: The area is a kite with dimensions h and c:

$$A(\alpha) = \frac{1}{2} \cdot h \cdot c$$

Let $\angle(h, a) = \beta$ and let x be the varying upper line segment:

$$\beta = \left(\frac{\pi}{2} - \alpha\right) \cdot \frac{1}{2} = \frac{\pi}{4} - \frac{\alpha}{2}$$

$$x = a \cdot \tan \beta$$

Then we have:

$$\begin{aligned} A(\alpha) &= \frac{1}{2} \cdot \sqrt{a^2 + x^2} \cdot 2a \sin \beta \\ &= a^2 \cdot \sqrt{1 + \tan^2 \beta} \cdot \sin \beta \\ &= a^2 \cdot \sqrt{1 + \tan^2 \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} \cdot \sin \left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \end{aligned}$$

Problem C.1

For this problem, we define the fractional part of $x \in \mathbb{R}_{\geq 0}$ as

$$\{x\} = x - \lfloor x \rfloor$$

where $\lfloor x \rfloor$ is the integer part of x, i.e., the greatest integer less than or equal to x.

(a) Draw the function $\{x\}$ in a coordinate system for $0 \leq x \leq 3$.

(b) Find the area A_n under the graph of $\{x\}$ between 0 and $n \in \mathbb{N}$ as given by:

$$A_n = \int_0^n \{x\}\, dx$$

Remember the definition of H_n from problem B.1. H_n grows similar to $\log(n)$ and they define the well-known constant γ in mathematics:

$$\gamma = \lim_{n\to\infty} (H_n - \log(n)) = 0.5772...$$

(c) Use this to prove the following identity:

$$\int_1^\infty \frac{\{x\}}{x^2}\, dx = 1 - \gamma$$

Hint: Split the integral into individual sums for each integer value.

Solution a: (sawtooth function from 0 to 3; three peaks)

Solution b:

$$A_n = \int_0^n \{x\}\ dx = n \cdot \int_0^1 x\ dx = n \cdot \left[\frac{x^2}{2}\right]_0^1 = \frac{n}{2}$$

Solution c:

$$\begin{aligned}
\int_1^\infty \frac{\{x\}}{x^2}\ dx &= \lim_{n\leftarrow\infty} \left[\int_1^2 \frac{x-1}{x^2}\ dx + \int_2^3 \frac{x-2}{x^2}\ dx + ... + \int_n^{n-1} \frac{x-(n-1)}{x^2}\ dx\right] \\
&= \lim_{n\leftarrow\infty} \left[\int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right)\ dx + \int_2^3 \left(\frac{1}{x} - 2\frac{1}{x^2}\right)\ dx + ... + \int_n^{n-1} \left(\frac{1}{x} - (n-1)\frac{1}{x^2}\right)\ dx\right] \\
&= \lim_{n\leftarrow\infty} \left[\int_1^n \frac{1}{x}\ dx - \int_1^2 \frac{1}{x^2}\ dx - 2\int_2^3 \frac{1}{x^2}\ dx - ... - (n-1)\int_n^{n-1} \frac{1}{x^2}\ dx\right] \\
&= \lim_{n\leftarrow\infty} \left[[\log(x)]_1^n - \left[-\frac{1}{x}\right]_1^2 - 2\left[-\frac{1}{x}\right]_2^3 - ... - (n-1)\left[-\frac{1}{x}\right]_{n-1}^n\right] \\
&= \lim_{n\leftarrow\infty} \left[\log(n) - \left(1 - \frac{1}{2}\right) - 2\left(\frac{1}{2} - \frac{1}{3}\right) - ... - (n-1)\left(\frac{1}{n-1} - \frac{1}{n}\right)\right] \\
&= \lim_{n\leftarrow\infty} \left[\log(n) - \sum_{k=2}^n (k-1)\left(\frac{1}{k-1} - \frac{1}{k}\right)\right] \\
&= \lim_{n\leftarrow\infty} \left[\log(n) - \sum_{k=2}^n \left(1 - \frac{k-1}{k}\right)\right] \\
&= \lim_{n\leftarrow\infty} \left[\log(n) - \sum_{k=2}^n \frac{1}{k}\right] \\
&= \lim_{n\leftarrow\infty} [\log(n) - H_n + 1] \\
&= 1 - \gamma
\end{aligned}$$

Problem C.2

This problem requires you to read following scientific article:

Sum of Reciprocals of Germain Primes.
Wagstaff, Samuel S. *Journal of Integer Sequences*, 24 (2021).
Link: https://cs.uwaterloo.ca/journals/JIS/VOL24/Wagstaff/wag4.pdf

Use the content of the article to work on the problems (a-f) below:
(a) What is the difference between twin primes and Germain primes? Give examples for both.

$\longrightarrow$ p is a twin prime iff $p+2$ or $p-2$ is also prime; Examples: 3, 5, 7, 11, 13, 17, 19
$\longrightarrow$ p is a Germain prime iff $2p+1$ is also prime; Examples: 2, 3, 5, 11, 23, 29, 41, 53

(b) Which numbers does the set $\mathcal{S}_{1,0}$ represent and what is the value of $S'_{1,2}(4 \cdot 10^{18})$?

$\longrightarrow$ $\mathcal{S}_{1,0} = \{p \ : \ p \text{ prime}\}$, i.e., the set of all prime numbers
$\longrightarrow$ from the first paragraph: $S'_{1,2}(4 \cdot 10^{18}) = 1.840503$

(c) In the proof of Theorem 1, explain why $\sum\limits_{p \le x, p \in \mathcal{S}_{a,b}} \frac{1}{p} = \sum\limits_{t=1}^{x} \frac{\pi_{a,b}(t) - \pi_{a,b}(t-1)}{t}$?

$\longrightarrow$ $\pi_{a,b}(t) - \pi_{a,b}(t-1)$ is 1 if $t \in \mathcal{S}_{a,b}$ (to increase $\pi_{a,b}(t)$ by one) and 0 otherwise; thus, the $1/t$ terms that are exactly $1/p$ with $p \in \mathcal{S}_{a,b}$ remain in the sum

(d) Explain the difference between Table 1 and Table 3.

$\longrightarrow$ Table 1 are the numerically calculated values $S_{a,b}(x)$ (up to x)
$\longrightarrow$ Table 3 shown an extended estimate by applying the Hardy-Littlewood approximation; the values are calculated with $S_{a,b}(x) + 2c_2/\log(x)$

(e) Use Theorem 3 to calculate an upper bound for $\pi_{1,16}(e^{100})$ in orders of magnitude.

$\longrightarrow$ $\pi_{1,16}(e^{100}) < \frac{16c_2e^{100}}{\log(e^{100})(8.37+\log(e^{100}))} = \frac{16c_2}{100\cdot(8.37+100)} \cdot e^{100} = \frac{16c_2}{100\cdot(8.37+100)} \cdot 10^{100\cdot\lg(e)} < 9.75 \cdot 10^{-4} \cdot 10^{43.5} < 10^{41}$

(f) Show in detail why the left- and right-hand side of equation (1) in Theorem 4 are equal.

$\longrightarrow$ For the integer domain it is $\pi'(t) = \pi(t) - \pi(t-1)$; thus, with integration by parts:

$$\int_M^N \frac{\pi(t)}{t^2}dt = \left[-\frac{\pi(t)}{t}\right]_M^N + \int_M^N \frac{\pi'(t)}{t}dt = \frac{\pi(M)}{M} - \frac{\pi(N)}{N} + \sum_{t=M}^{N} \frac{\pi(t) - \pi(t-1)}{t}dt$$