Pre-Final Round 2020

EXAMPLE SOLUTION

Problem A.1

Find all points (x, y) where the functions $f(x), g(x), h(x)$ have the same value:

$$f(x) = 2^{x-5} + 3, \quad g(x) = 2x - 5, \quad h(x) = \frac{8}{x} + 10$$

Step 1: $g(x) = h(x) \implies 0 = 2x^2 - 15x - 8 \implies x \in \{-0.5, 8\}$

Step 2: $f(-0.5) \neq g(-0.5),\ f(8) = g(8) = 11$

Solution: $(8, 11)$

Problem A.2

Determine the roots of the function $f(x) = (5^{2x} - 6)^2 - (5^{2x} - 6) - 12$.

Substitute $z = 5^{2x} - 6$:

$$0 = z^2 - z - 12 \implies z \in \{-3, 4\}$$

Solve for x:

$$z = 5^{2x} - 6 = 25^x - 6 \implies x = \log_{25}(z + 6)$$

Solution: $x = \log_{25}(3) \approx 0.341$ and $x = \log_{25}(10) \approx 0.715$

Problem A.3

Find the derivative $f_m'(x)$ of the following function with respect to x:

$$f_m(x) = \left(\sum_{n=1}^{m} n^x \cdot x^n \right)^2$$

$$f(x) = 2 \cdot \left(\sum_{n=1}^{m} n^x \cdot x^n \right) \cdot \left[\sum_{n=1}^{m} n^x \cdot x^n \right]' = 2 \cdot \left(\sum_{n=1}^{m} n^x \cdot x^n \right) \cdot \left(\sum_{n=1}^{m} n^x \cdot x^n \cdot \left(\frac{n}{x} + \log n \right) \right)$$

Problem A.4

Find at least one solution to the following equation:

$$\frac{\sin(x^2-1)}{1-\sin(x^2-1)} = \sin(x) + \sin^2(x) + \sin^3(x) + \sin^4(x) + \cdots$$

Right-hand side (with $|\sin(x)| < 1$):

$$\sin(x) + \sin^2(x) + \cdots = \sin(x) \cdot \big(1 + \sin(x) + \sin^2(x) + \cdots\big) = \frac{\sin(x)}{1-\sin(x)}$$

It follows:

$$\sin(x^2-1) = \sin(x) \implies x^2 - 1 = x \implies 0 = x^2 - x - 1 \implies x \in \left\{ \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right\}$$

Solution: $x = \frac{1-\sqrt{5}}{2} \approx -0.618$ (Alternative solutions possible.)

Problem B.1

Consider the following sequence of successive numbers of the 2^k-th power:

$$1,\ 2^{2^k},\ 3^{2^k},\ 4^{2^k},\ 5^{2^k},\ ...$$

Show that the difference between the numbers in this sequence is odd for all $k \in \mathbb{N}$.

The difference Δ_n can be written as:

$$\Delta_n = (n+1)^{2^k} - n^{2^k}$$

If $n \equiv 0 \pmod 2$:

$$\Delta_n \equiv (0+1)^{2^k} - 0^{2^k} = 1^{2^k} = 1 \pmod 2$$

If $n \equiv 1 \pmod 2$:

$$\Delta_n \equiv (1+1)^{2^k} - 1^{2^k} \equiv 0^{2^k} - 1^{2^k} = -1 \equiv 1 \pmod 2$$

Alternative: Proof by induction.

Problem B.2

Prove this identity between two infinite sums (with $x \in \mathbb{R}$ and $n!$ stands for factorial):

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

By using the series expansion of the exponential function $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$:

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = (e^x)^2 = e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Alternative: By using the fact that $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n \;\Rightarrow\; \sum\limits_{k=0}^{n} \frac{1}{k!(n-k)!} = \frac{2^n}{n!}$ (must be proven):

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^{n+m}}{n!m!} = \sum_{N=0}^{\infty} x^N \sum_{k=0}^{N} \frac{1}{k!(N-k)!} = \sum_{N=0}^{\infty} \frac{2^N x^N}{N!}$$

Problem B.3

You have given a function $\lambda : \mathbb{R} \to \mathbb{R}$ with the following properties ($x \in \mathbb{R},\ n \in \mathbb{N}$):

$$\lambda(n) = 0\ , \quad \lambda(x+1) = \lambda(x)\ , \quad \lambda\left(n + \frac{1}{2}\right) = 1$$

Find two functions $p, q : \mathbb{R} \to \mathbb{R}$ with $q(x) \neq 0$ for all $x \in \mathbb{R}$ such that $\lambda(x) = q(x)(p(x)+1)$.

From $\lambda(n) = 0$ and $q(x) \neq 0$ it follows:

$$p(n) = -1$$

Because of $\lambda(x+1) = \lambda(x)$, we set:

$$p(x+1) = p(x),\ q(x+1) = q(x)$$

Consider the periodic function $p(x) = \sin(...)$ with period $T = 1$ and $p(n) = -1$:

$$p(x) = \sin\left(2\pi x - \frac{\pi}{2}\right)$$

From $\lambda\left(n + \frac{1}{2}\right) = 1$ it follows:

$$q\left(\frac{1}{2}\right) = \frac{1}{p\left(\frac{1}{2}\right) + 1} = \frac{1}{2}$$

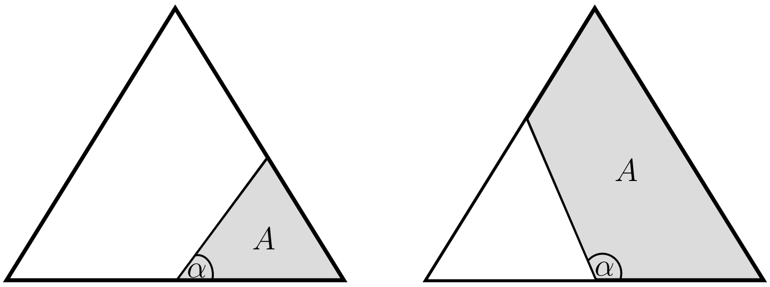
This satisfies $q(x+1) = q(x) = \frac{1}{2}$ and $q(x) \neq 0$. Thus:

$$p(x) = \sin\left(2\pi x - \frac{\pi}{2}\right), \quad q(x) = \frac{1}{2}$$

(Alternative solutions possible.)

Problem B.4

You have given an equal sided triangle with side length a. A straight line connects the center of the bottom side to the border of the triangle with an angle of α. Derive an expression for the enclosed area $A(\alpha)$ with respect to the angle (see drawing).



First we consider the area $\tilde{A}(\alpha)$ for $\alpha \leq \frac{\pi}{2}$: The gray area triangle has the following two angles:

$$\alpha, \quad \beta = 60° = \frac{\pi}{3}$$

The third angle follows:

$$\gamma = 180° - 60° - \alpha = \pi - \frac{\pi}{3} - \alpha = \frac{2\pi}{3} - \alpha,$$

The length of the connecting line c follows from the sine rule:

$$\frac{\sin\left(\frac{\pi}{3}\right)}{c} = \frac{\sin(\gamma)}{a/2} \implies c = \frac{a}{2}\frac{\sin\left(\frac{\pi}{3}\right)}{\sin(\gamma)}$$

The height of the gray triangle is $h = c \cdot \sin(\alpha)$. Thus:

$$\tilde{A}(\alpha) = \frac{1}{2} \cdot \frac{a}{2} \cdot h = \frac{1}{2} \cdot \frac{a}{2} \cdot \frac{a}{2}\frac{\sin\left(\frac{\pi}{3}\right)}{\sin(\gamma)}\sin(\alpha) = \frac{a^2}{8} \cdot \frac{\sin\left(\frac{\pi}{3}\right)}{\sin\left(\frac{2\pi}{3} - \alpha\right)}\sin(\alpha)$$

And with $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$:

$$\tilde{A}(\alpha) = \frac{\sqrt{3}}{16} \cdot \frac{\sin(\alpha)}{\sin\left(\frac{2\pi}{3} - \alpha\right)} \cdot a^2$$

The area at $\alpha = \frac{\pi}{2}$ is then equal to $\tilde{A}\left(\frac{\pi}{2}\right) = \frac{\sqrt{3}}{8}a^2$. For all α we find:

$$A(\alpha) = \begin{cases} \tilde{A}(\alpha) & \alpha \leq \frac{\pi}{2} \\ \tilde{A}\left(\frac{\pi}{2}\right) - \tilde{A}(\pi - \alpha) & \alpha > \frac{\pi}{2} \end{cases} = \begin{cases} \tilde{A}(\alpha) & \alpha \leq \frac{\pi}{2} \\ \frac{\sqrt{3}}{8}a^2 - \tilde{A}(\pi - \alpha) & \alpha > \frac{\pi}{2} \end{cases}$$

Problem C.1

Let $\pi(N)$ be the number of primes less than or equal to N (example: $\pi(100) = 25$). The famous prime number theorem then states (with $\sim$ meaning *asymptotically equal*):

$$\pi(N) \sim \frac{N}{\log(N)}$$

Proving this theorem is very hard. However, we can derive a statistical form of the prime number theorem. For this, we consider *random primes* which are generated as follows:

(i) Create a list of consecutive integers from 2 to N.

(ii) Start with 2 and mark every number > 2 with a probability of $\frac{1}{2}$.

(iii) Let n be the next non-marked number. Mark every number $> n$ with a probability of $\frac{1}{n}$.

(iv) Repeat (iii) until you have reached N.

All the non-marked numbers in the list are called *random primes*.

(a) Let q_n be the probability of n being selected as a *random prime* during this algorithm. Find an expression for q_n in terms of q_{n-1}.

(b) Prove the following inequality of q_n and q_{n+1}:

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}$$

(c) Use the result from (b) to show this inequality:

$$\sum_{k=1}^{N} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1$$

(d) With this result, derive an asymptotic expression for q_n in terms of n.

(e) Let $\tilde{\pi}(N)$ be the number of *random primes* less than or equal to N. Use the result from (d) to derive an asymptotic expression for $\tilde{\pi}(N)$, i.e. the prime number theorem for *random primes*.

Solution (a): q_{n-1} is the probability of n being not marked before $n-1$; $\frac{1}{n-1}$ is the probability of n being marked if $n-1$ is marked (with q_{n-1} probability):

$$q_n = q_{n-1}\left(1 - \frac{q_{n-1}}{n-1}\right)$$

Solution (b): From (a) we have:

$$q_{n+1} = q_n\left(1 - \frac{q_n}{n}\right) \quad \Rightarrow \quad \frac{1}{q_{n+1}} = \frac{1}{q_n} + \frac{1}{n - q_n}$$

Since $0 < q_n \leq 1$:

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}$$

Solution (c): Rearranging the inequality:

$$\frac{1}{n} < \frac{1}{q_{n+1}} - \frac{1}{q_n} < \frac{1}{n-1}$$

We sum over the inequality from $n = 3$ to $N - 1$:

$$\sum_{n=3}^{N-1} \frac{1}{n} < \sum_{n=3}^{N-1} \left(\frac{1}{q_{n+1}} - \frac{1}{q_n} \right) < \sum_{n=3}^{N-1} \frac{1}{n-1}$$

For the middle term (with $q_2 = 1$ and thus $q_3 = 1/2$):

$$\sum_{n=3}^{N-1} \left(\frac{1}{q_{n+1}} - \frac{1}{q_n} \right) = \frac{1}{q_N} - \frac{1}{q_3} = \frac{1}{q_N} - 2$$

Adding $+2$ to the left-hand side:

$$\sum_{n=3}^{N-1} \frac{1}{n} + 2 = \sum_{n=1}^{N-1} \frac{1}{n} + \frac{1}{2} > \sum_{n=1}^{N} \frac{1}{n}$$

Adding $+2$ to the right-hand side:

$$\sum_{n=3}^{N-1} \frac{1}{n-1} + 2 = \sum_{n=2}^{N-2} \frac{1}{n} + 2 = \sum_{n=1}^{N-2} \frac{1}{n} + 1 < \sum_{n=1}^{N} \frac{1}{n} + 1$$

This gives:

$$\sum_{n=1}^{N} \frac{1}{n} < \frac{1}{q_N} < \sum_{n=1}^{N} \frac{1}{n} + 1$$

Solution (d): It is well-know that the harmonic series grows like $\log(N)$, i.e.:

$$\sum_{n=1}^{N} \frac{1}{n} \sim \log(N)$$

With the inequality from (c) then follows:

$$\frac{1}{q_N} \sim \log(N) \implies q_N \sim \frac{1}{\log(N)}$$

Solution (e): For $\tilde{\pi}(N)$ we find the following (i.e. the expectation value):

$$\tilde{\pi}(N) = \sum_{n=2}^{N} q_n \sim \sum_{n=2}^{N} \frac{1}{\log(n)} \sim \frac{N}{\log(N)}$$

Problem C.2

This problem requires you to read following scientific article:

On the harmonic and hyperharmonic Fibonacci numbers.
Tuglu, N., Kızılateş, C. & Kesim, S. Adv Differ Equ (2015).
Link: https://doi.org/10.1186/s13662-015-0635-z

Use the content of the article to work on the problems (a-f) below. All problems marked with * are bonus problems (g-i) that can give you extra points. However, it is not possible to get more than 40 points in total.

Solution (a) What are the values of H_n, F_n and $\mathbb{F}_n$ for $n = 1, 2, 3$?

$H_1 = 1, \quad H_2 = \frac{3}{2} = 1.5, \quad H_3 = \frac{11}{6} = 1.8\overline{3}$
$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2$
$\mathbb{F}_1 = 1, \quad \mathbb{F}_2 = 2, \quad \mathbb{F}_3 = \frac{5}{2} = 2.5$

Solution (b) Determine the hyperharmonic number $H_3^{(10)}$ (Tip: use Equation 4) and $F_2^{(3)}$.

Applying Equation 4: $H_3^{(10)} = \sum_{t=1}^{3} \binom{3+10-t-1}{10-1} \frac{1}{t} = \sum_{t=1}^{3} \binom{12-t}{9} \frac{1}{t} = \frac{181}{3} = 60.\overline{3}$

From the definition: $F_2^{(3)} = \sum_{k=0}^{2} \sum_{i=0}^{k} F_i = F_0 + (F_0 + F_1) + (F_0 + F_1 + F_2) = 3$

Solution (c) Use the definition of $x^{\underline{m}}$ to simplify the following fraction: $\frac{x^{\underline{m+1}} - x^{\underline{m}}}{x^{\underline{m}} + x^{\underline{m+1}}}$

$$\frac{x^{\underline{m+1}} - x^{\underline{m}}}{x^{\underline{m}} + x^{\underline{m+1}}} = \frac{(x-m)-1}{1+(x-m)} = \frac{x-m-1}{x-m+1}$$

Solution (d) Present the proof of Theorem 1 step-by-step by applying Equation 6.

We set

$$u(k) = \mathbb{F}_k, \quad \Delta v(k) = 1$$

such that

$$\Delta u(k) = \mathbb{F}_{k+1} - \mathbb{F}_k = \frac{1}{F_{k+1}}, \quad v(k) = k$$

With Equation 6 we get:

$$\sum_{k=0}^{n-1} \mathbb{F}_k = \sum_{k=0}^{n-1} \mathbb{F}_k \cdot 1 = k\mathbb{F}_k\big|_0^{n-1+1} - \sum_{k=0}^{n-1} Ev(k)\frac{1}{F_{k+1}} = k\mathbb{F}_k\big|_0^{n} - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}} = n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}}$$

Solution (e) Show that $\mathbb{F}_n^{(r)} - \mathbb{F}_{n-2}^{(r)} = \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)}$.

$$\mathbb{F}_n^{(r)} = \sum_{k=1}^{n} \mathbb{F}_k^{(r-1)} = \sum_{k=1}^{n-2} \mathbb{F}_k^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)} + \mathbb{F}_n^{(r-1)} = \mathbb{F}_{n-2}^{(r)} + \mathbb{F}_{n-1}^{(r-1)} + \mathbb{F}_n^{(r-1)}$$

Solution (f) Determine the Euclidean norm of the circulant matrix Circ(1, 1, 0, 0).

$$\text{Circ}(1,1,0,0) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \implies \|A\|_E = \sqrt{8} \approx 2.83$$

Solution (g*) Show that for $u(k) = \mathbb{F}_k^2$ we get $\Delta u(k) = \frac{1}{F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$.

$$\begin{aligned}\Delta u(k) = \mathbb{F}_{k+1}^2 - \mathbb{F}_k^2 &= \sum_{i \text{ or } j=k+1} \frac{1}{F_i F_j} = \sum_{i \text{ xor } j=k+1} \frac{1}{F_i F_j} + \frac{1}{F_{k+1}^2} \\ &= 2\sum_i \frac{1}{F_i F_{k+1}} + \frac{1}{F_{k+1}^2} = \frac{2\mathbb{F}_k}{F_{k+1}} + \frac{1}{F_{k+1}^2} = \frac{1}{F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)\end{aligned}$$

Solution (h*) Use the theorems from the article to prove the following identity:

$$\sum_{k=1}^{n-1} k^{\underline{m}} (\mathbb{F}_k)^2 = \frac{n^{\underline{m+1}}}{m+1}\mathbb{F}_n^2 - \sum_{k=0}^{n-1} \frac{(k+1)^{\underline{m+1}}}{(m+1)F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$$

Same steps as in Theorem 4, except with $u(k) = \mathbb{F}_k^2$ (see Theorem 2).

Solution (i*) Use Equation 1 and Theorem 5 to show the following:

$$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right)$$

Rearranging Equation 1 and setting $k = 0$ gives:

$$H_n = 1 + \frac{1}{n}\sum_{k=0}^{n-1} H_k$$

Inserting into Theorem 5:

$$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = \mathbb{F}_n + \frac{\mathbb{F}_n}{n}\sum_{k=0}^{n-1} H_k - \sum_{k=0}^{n-1}\frac{H_{k+1}}{F_{k+1}} = \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right)$$