SUBIN MANANDHAR

I Y M C

Pre-Final Round 2018

DATE OF RELEASE: 29. OCTOBER 2018

Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend to print this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your own sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The 15 problems are separated into three categories: 6x basic problems (A; three points), 6x advanced problems (B; four points), 3x special-creativity problems (C; six points).
- You receive points for the correct solution as well as for for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 60 points in total. You qualify for the final round if you reach at least 25 points (under 18 years) or 35 points (over 18 years).
- The special-creativity problems do not require a final answer to the problem to give points. The approaches, interim results and discussions give between zero to six points. These problems may include real aspects of unsolved mathematical questions.
- Please consider following notation that is used for the problems
  - $x \in \mathbb{R}$ denotes a real number, $n \in \mathbb{N}$ denotes a positive integer.
  - f, g denote functions. (The domain and co-domain should follow from the context.)
  - The "roots" of a function f are those x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/status`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sun 4. November 2018, 23:59 UTC+0**.
- The results of the pre-final round will be announced on 12. November 2018.

Good luck!

Problem A.1

Find the roots of the function $f(x) = 2^3x^3 + 2^2x^2 + 2x + 1$.

$$f(x) = 2^3x^3 + 2^2x^2 + 2x + 1 = 0$$
$$2^2x^2(2x+1) + (2x+1) = 0$$
$$(2x+1)(2^2x^2+1) = 0$$

Either $2x+1=0$ OR $2^2x^2+1=0$

$x = \frac{-1}{2}$

$x^2 = -\frac{1}{2^2}$

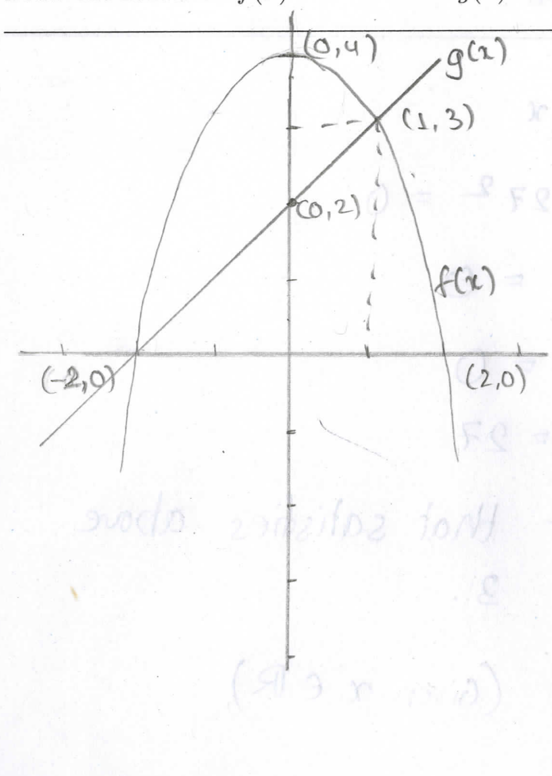
$x = \pm\sqrt{\frac{-1}{2^2}}$

$= \pm \frac{i}{2}$ where $i = \sqrt{-1}$

So, $\boxed{x = -\frac{1}{2} \text{ or } \pm\frac{i}{2}}$

Problem A.2

Draw the functions $f(x) = 4 - x^2$ and $g(x) = x + 2$ and find the points of intersection (x, y).



$f(x) = 4 - x^2$ forms a parabola opening downward, and passing through (0,4), (2,0) and (-2,0)

$g(x) = x+2$ is a straight line with slope = 1 and passing through (0,2), (-2,0), and (1,3).

The points of intersection are (1,3) and (-2,0).

They satisfy both functions.

Problem A.3

Find the derivative $f'(x)$ of the function $f(x) = 2^x \cdot x^2$.

$$f(x) = 2^x \cdot x^2$$

$$f'(x) = 2^x(2x) + x^2(2^x \cdot \ln(2))$$

$$\left[\begin{array}{l} \text{If } f(x) = u(x)\cdot v(x). \\ f'(x) = u(x)\cdot v'(x) \\ \quad + v(x)\cdot u'(x) \end{array} \right]$$

$$\boxed{f'(x) = 2\cdot 2^x \cdot x + 2^x \cdot x^2 \cdot \ln(2)}$$

Problem A.4

Determine all x that solve the equation $x^{2x} + 27^2 = 54x^x$.

$$x^{2x} + 27^2 = 54x^x$$

$$(x^x)^2 - 54x^x + 27^2 = 0$$

$$(x^x - 27)^2 = 0$$

$$x^x - 27 = 0$$

$$x^x = 27$$

The only real number that satisfies above equation is 3.

So, $x = 3$ (Given $x \in \mathbb{R}$)

Problem A.5

Find all x such that $|x^2 - 1| < 2x$.

$$|x^2 - 1| < 2x$$

$$-2x < x^2 - 1 < 2x$$

$$-2x < x^2 - 1$$
$$0 < x^2 + 2x - 1$$
$$2 < x^2 + 2x + 1$$
$$2 < (x+1)^2$$
$$\sqrt{2} < x + 1 \Rightarrow x > \sqrt{2} - 1$$

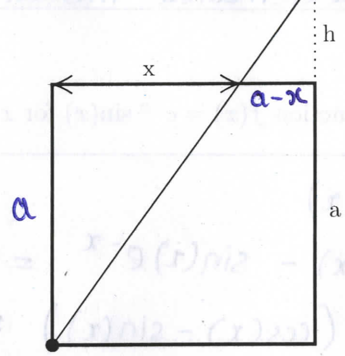
$$x^2 - 1 < 2x$$
$$x^2 - 2x + 1 < 2$$
$$(x-1)^2 < 2$$
$$x - 1 < \sqrt{2}$$
$$\Rightarrow x < \sqrt{2} + 1$$

So,

$$\boxed{\sqrt{2} - 1 < x < \sqrt{2} + 1}$$

Problem A.6

You have given a square with side a and an intersecting straight line in a distance of x as seen below. Find an equation for the height $h(a, x)$.



The upperright triangle and the upperleft triangle in the square are similar since the angles are the same.

For similar triangles, ratios of corresponding sides are equal:

$$\frac{h}{a} = \frac{a - x}{x}$$

$$\Rightarrow \boxed{h = a\left(\frac{a-x}{x}\right) = a\left(\frac{a}{x} - 1\right)} = \boxed{\frac{a^2}{x} - a}$$

Problem B.1

Show that $2^{3n} - 1$ is divisible by 7 for all positive integers n.

Solving using Mathematical Induction:

① For $n=1$, $2^{3n}-1 = 2^3 - 1 = 7$
which is clearly divisible by 7.

② Suppose for $n=k$, $2^{3k}-1$ is divisible by 7.

③ For $n=k+1$, $2^{3(k+1)}-1 = 2^{3k}.2^3 - 1 = 8.2^{3k} - 1$
$= 7.2^{3k} + 2^{3k} - 1$

Here, both 7.2^{3k} and $(2^{3k}-1)$ are divisible by 7.
So, $2^{3(k+1)}-1$ is divisible by 7.

Hence, by mathematical induction, $2^{3n}-1$ is divisible by 7 for all $n \in \mathbb{N}$

Problem B.2

Determine the biggest value of the function $f(x) = e^{-x}\sin(x)$ for $x \geq 0$.

$f(x) = e^{-x}\sin(x)$

$f'(x) = e^{-x}\cos(x) - \sin(x)e^{-x} = 0$

$e^{-x}(\cos(x) - \sin(x)) = 0$

" e^{-x} tends to 0 only when x tends to ∞."
So, neglecting this,

$\cos(x) - \sin(x) = 0$

$\cos(x) = \sin(x)$

$\tan(x) = \frac{\pi}{4} + n\pi$ [This gives both local maxima and minima]

Putting into $f(x)$,

$f(\frac{\pi}{4}+n\pi) = \frac{\sin(\frac{\pi}{4}+n\pi)}{e^{(\frac{\pi}{4}+n\pi)}}$

This value shrinks as we increase n.
So max value is at $n=0$ ($x \geq 0$)

At $n=0$,
$f(\frac{\pi}{4}) = \frac{\sin(\frac{\pi}{4})}{e^{\pi/4}} = \boxed{\frac{1}{\sqrt{2}.e^{\pi/4}}}$

Problem B.3

Find the value of this infinite sum: $\sum_{n=0}^{\infty} \frac{2^{2n}+2^n}{2^{3n}}$.

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{2^{3n}} + \sum_{n=0}^{\infty} \frac{2^n}{2^{3n}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} + \sum_{n=0}^{\infty} \frac{1}{2^{2n}}$$

$a = 1 \quad r = \frac{1}{2}$; $a = 1 \quad r = \frac{1}{4}$

$\text{Sum} = \frac{1}{1-1/2} = 2$; $\text{Sum} = \frac{1}{1-1/4} = 4/3$

$$= 2 + \frac{4}{3}$$

$$= \boxed{\frac{10}{3}}$$

Problem B.4

Give a closed expression for the function $g(n)$ with the following behaviour:

$$g(n) = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

We can use the property that $(-1)^n$ changes sign as n changes from even to odd or vice versa.

So, $g(n) = \frac{n}{2}\left(1-(-1)^n\right)$

When n is even, $g(n) = \frac{n}{2}(1-1) = 0$

When n is odd, $g(n) = \frac{n}{2}(1+1) = n$

Problem B.5

Find a function $\omega(x)$ such that the function $f(x) = \sin(\omega(x))$ has the roots at $\pi, \pi^2, \pi^3, \ldots$.

$f(x) = \sin(\omega(x)) = 0$

$\Rightarrow \omega(x) = 0, \pi, 2\pi, \cdots n\pi$ (n is an integer)

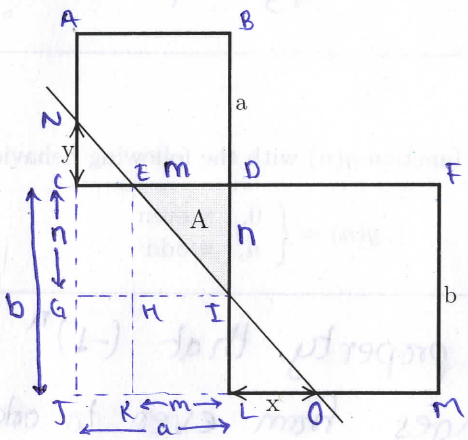
We need a function that brings powers out front as a coefficient, and leave only π.

$\boxed{\omega(x) = \log_{\sqrt[\pi]{\pi}}(x)}$ satisfies this condition..

$\omega(\pi^n) = \log_{\sqrt[\pi]{\pi}}(\pi^n) = n \log_{\sqrt[\pi]{\pi}}(\pi) = n\pi$

Problem B.6

The drawing below shows two squares with side a and b. A straight line intersects the squares at y and x (see drawing). Calculate the gray area $A(a, b, x, y)$ between the squares and the line.



ΔNIO and ΔNGI are similar and so are ΔEKO and ΔNJO.

So, $\dfrac{NJ}{NG} = \dfrac{JO}{GI}$

$\dfrac{b+y}{n+y} = \dfrac{a+x}{a}$

$n+y = \left(\dfrac{b+y}{a+x}\right)a$

$n = \left(\dfrac{b+y}{a+x}\right)a - y$

So, $\dfrac{NJ}{EK} = \dfrac{JO}{KO}$

$\dfrac{b+y}{b} = \dfrac{a+x}{m+x}$

$m+x = \left(\dfrac{a+x}{b+y}\right)b$

$m = \left(\dfrac{a+x}{b+y}\right)b - x$

So, area $= \dfrac{1}{2}mn = \dfrac{1}{2}\left[\left(\left(\dfrac{a+b}{b+y}\right)b - x\right)\left(\left(\dfrac{b+y}{a+x}\right)a - y\right)\right]$