# IYMC Pre-Final Round

FLORES, Richell Isaiah S.

Problem A-1.

Completely factoring $f(x)$ gives,

$$f(x) = (2^3x^3 + 2x) + (2^2x^2 + 1)$$
$$= 2x\,(2^2x^2+1) + (2^2x^2+1)$$
$$= (2^2x^2+1)(2x+1)$$

Equating $f(x)$ to 0 to identify the roots,

$$f(x) = 0$$
$$(2^2x^2+1)(2x+1) = 0$$

$$2^2x^2 + 1 = 0 \qquad \text{or} \qquad 2x+1 = 0$$
$$4x^2 + 1 = 0 \qquad\qquad 2x = -1$$
$$4x^2 = -1 \qquad\qquad x = -\tfrac{1}{2}$$
$$x^2 = -\tfrac{1}{4}$$
$$x = \pm\tfrac{1}{2}i$$

$\therefore$ the roots of $f(x)$ are $x = \pm\frac{1}{2}i,\ -\frac{1}{2}$.

Problem A.2.

The point of intersection $(x, y)$ is the point where the graphs have the same $x$- and $y$-coordinates. It can also be interpreted as the solution to a system of equations given by the two graphs. Constructing the system,

$$\begin{cases} y = 4 - x^2 & \cdots (1) \\ y = x + 2 & \cdots (2) \end{cases}$$

Substitute (1) into (2)

$$4 - x^2 = x + 2$$

Solving for $x$,

$$-x^2 - x + 2 = 0$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2,\ 1$$

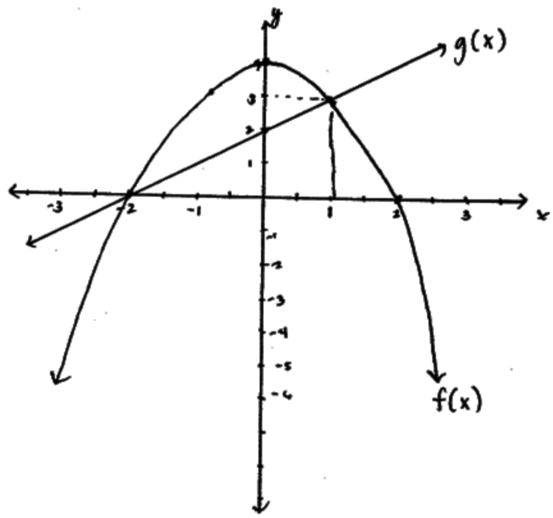
Substitute to (2)

$$y = (-2) + 2 = 0 \quad \text{when } x = -2$$
$$y = (1) + 2 = 3 \quad \text{when } x = 1$$

$\therefore$ the point of intersection are

$$(-2, 0) \quad \text{and} \quad (1, 3)$$

$f(x)$ is a parabola facing downward similar to the basic shape of $y = -x^2$ but translated up 4 units along the $y$-axis. $g(x)$ is a line similar to the basic shape of $y = x$ but translated 2 units up along the $y$-axis. Graphing,



Problem A.3.

To find $f'(x)$, use the Product Rule.

$$f(x) = 2^x \cdot x^2$$

$$f'(x) = (2^x)' \cdot x^2 + 2^x (x^2)'$$

$$= 2^x \ln 2 \cdot x^2 + 2^x \cdot 2x$$

$$= x 2^x (x \ln 2 + 2)$$

Problem A.4.

Given the equation

$$x^{2x} + 27^2 = 54 x^x$$

Rearranging,

$$x^{2x} - 54 x^x + 27^2 = 0$$

$$(x^x)^2 - 54x^x + 27^2 = 0 \qquad \leftarrow \text{a perfect square trinomial}$$

Solving,

$$(x^x - 27)^2 = 0$$

So,

$$x^x - 27 = 0$$

$$x^x = 27$$

Expressing RHS as a product of primes gives

$$x^x = 3^3$$

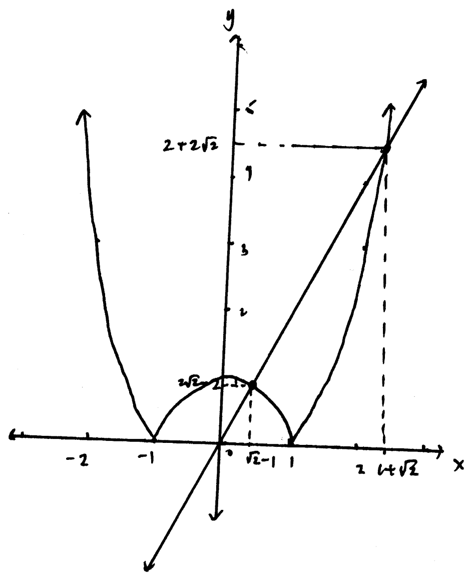
$$\therefore \; x = 3$$

Problem A-5.

The problem could be interpreted as finding the interval for which the line $y = 2x$ is above the graph $y = |x^2 - 1|$.

The graph of $y = |x^2-1|$ is given by:

$$y = \begin{cases} x^2 - 1, & x \in (-\infty, -1] \cup [1, +\infty) \\ -x^2+1, & x \in (-1, 1) \end{cases}$$




Finding the points of intersection for both cases:

when $y = x^2 - 1$,

$$x^2 - 1 = 2x$$
$$x^2 - 2x - 1 = 0$$
$$x = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$$
$$= 1 \pm \sqrt{2}$$

when $y = -x^2 + 1$,

$$-x^2 + 1 = 2x$$
$$-x^2 - 2x + 1 = 0$$
$$x^2 + 2x - 1 = 0$$
$$x = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$
$$= -1 \pm \sqrt{2}$$

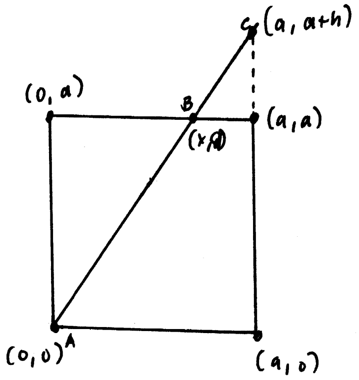
From the graph, the valid points of intersection are at

$x = \sqrt{2} - 1$ and $x = 1 + \sqrt{2}$

$\therefore \quad x \in (\sqrt{2} - 1, 1 + \sqrt{2})$

Problem A.6.

Labeling the coordinates of the important points of the figure as follows:




Since points A, B, and C lie on the same line, they should have the same slope. Expressed mathematically,

$$\frac{a}{x} = \frac{a+h}{a} = \frac{h}{a-x} \qquad (1)$$

From (1), the following equations could be obtained:

$$a^2 = x(a+h) \Rightarrow a^2 = ax + hx \qquad (2)$$

$$a(a-x) = xh \Rightarrow a^2 - ax = xh \qquad (3)$$

Since (2) & (3) are equivalent, solve h from either equations.

$$a^2 = ax + hx$$

$$a^2 - ax = hx$$

$$h = \frac{a^2 - ax}{x}$$

$$\therefore\ h(a,x) = \frac{a^2 - ax}{x}$$

Problem B.1.

the proof will utilize induction.

① check if $2^{3n}-1$ is divisible by 7 for $n=1$

$$2^{3n}-1 = 2^{3(1)}-1$$

$$= 8-1 = 7$$

7 is divisible by 7.

② Assume that the statement is true for $n=k$. so

$$2^{3k}-1 \equiv 0 \pmod 7$$

$$\therefore\ 2^{3k}-1 = 7m \quad \text{for } m \in \mathbb{Z}^+$$

$$\Rightarrow 2^{3k} = 7m+1$$

③ Show that the statement holds true for $n=k+1$

$$2^{3(k+1)}-1 = 2^{3k+3}-1$$

$$= 2^{3k}\cdot 2^{3}-1$$

$$= (7m+1)(8)-1$$

$$= 56m+8-1 = 56m+7 = 7\overbrace{(8m+1)}^{\in \mathbb{Z}^+}$$

Since $2^{3n}-1$ could be expressed as a product of 7 and an integer, and holds true for the steps in the induction, $2^{3n}-1$ is indeed divisible by 7.

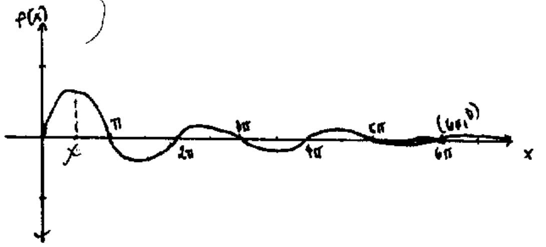
Problem B.2.

Based from f(x), it can be clearly seen that the function is sinusoidal. However it has a factor of $e^{-x}$ that determines the amplitude of the graph. Since $e^{-x}$ is also dependent on x, the graph is expected to resemble that of a damped oscillation. Moreover, it is worth noting that as $x \to +\infty$, $e^{-x} \to 0$. Following is the reasoning for the said claim:

Since $e^{-x}$ is also equivalent to $\frac{1}{e^x}$, and as $x \to +\infty$, $e^x$ increases without bound. Since it is the denominator, the whole expression $\frac{1}{e^x}$ approaches 0. Expressed mathematically,

$$\lim_{x \to +\infty} e^{-x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

With this behavior, the sketch of f(x) for $x \geq 0$ could be obtained as follows.




From the sketch, the maxima could be found in the interval $[0, \pi]$. Creating a variation table for this interval,

$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$$
$$= e^{-x}(\cos x - \sin x)$$

$$f'(x) = \overbrace{e^{-x}}^{\text{cannot be zero}}(\cos x - \sin x)$$

so $\cos x - \sin x = 0$
$\cos x = \sin x \Rightarrow x = \frac{\pi}{4}$

| $x$ | $0$ | $\cdots$ | $\frac{\pi}{4}$ | $\cdots$ | $\pi$ |
|---|---|---|---|---|---|
| $f(x)$ | $0$ | | $\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}$ | | $0$ |
| $f'(x)$ | | $+$ | $0$ | $-$ | |
| | | $\nearrow$ | max | $\searrow$ | |

$\therefore$ the biggest value of f(x) is $\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}$

Problem B. 3.

The sum could be expressed as:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{1}{2^{2n}}\right)$$

Using rules or properties of summations, this is equivalent to:

$$\sum_{n=0}^{\infty}\frac{1}{2^n} + \sum_{n=0}^{\infty}\frac{1}{2^{2n}}$$

Expanding this notation gives:

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots\right)+\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^8}+\cdots\right)$$

Observe that $\sum_{n=0}^{\infty}\frac{1}{2^n}$ forms an infinite geometric series with first term 1 and common ratio $\frac{1}{2}$.

Also, $\sum_{n=0}^{\infty}\frac{1}{2^{2n}}$ forms an infinite geometric series with first term 1 and common ratio $\frac{1}{2^2}$.

Using $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ (for $|r|<1$), the sum could be calculated.

$$\sum_{n=0}^{\infty}\frac{1}{2^n} = \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}} = \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{2^2}\right)^n = \frac{1}{1-\frac{1}{2^2}} = \frac{4}{3}$$

$$\therefore \sum_{n=0}^{\infty}\frac{2^{2n}+2^n}{2^{3n}} = \sum_{n=0}^{\infty}\frac{1}{2^n}+\sum_{n=0}^{\infty}\frac{1}{2^{2n}} = 2+\frac{4}{3} = \frac{10}{3}$$

Problem B.4.

Expressing the first terms of $g(n)$:

$$0, 1, 0, 3, 0, 5, 0, 7, 0, 9, 0, 11 \cdots$$

Observe that the sequence oscillates between 0 and a non-zero number equivalent to its position. This oscillation gives the idea that the closed expression includes the function $\sin x$. However, $x$ should be expresseds as a term that could make $\sin x$ produce integer values for integer values of $n$. Also, these integer values should be 0, -1, or 1 for integer values of $n$. A function that fits this criteria is

$$f(n) = \sin\left(\frac{n}{2}\pi\right) \qquad (n = 0, 1, 2, 3, \ldots)$$

Looking at the first values of $f(n)$,

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $f(n)$ | 0 | 1 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | |

Multiplying $n$ to $f(n)$ gives,

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n \cdot f(n)$ | 0 | 1 | 0 | -3 | 0 | 5 | 0 | -7 | 0 | ... |

Since there are negative values, putting the entire expression in an absolute value gives,

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert n \cdot f(n) \rvert$ | 0 | 1 | 0 | 3 | 0 | 5 | 0 | 7 | 0 | ... |

which is exactly what the problem requires.

Therefore,

$$g(n) = \left| n \cdot \sin\left(\frac{n\pi}{2}\right) \right| \qquad (n = 0, 1, 2, 3, 4, \ldots)$$

Problem B.5.

Firstly, the roots are $\pi, \pi^2, \pi^3, \ldots$, so $w(x)$ should produce these values. Also, since the roots happen when $f(x) = \sin(w(x)) = 0$, $w(x) = 0, \pi, 2\pi, \cdots$ The following table summarizes this information.

| $x$ | $\pi$ | $\pi^2$ | $\pi^3$ | $\pi^4$ | $\cdots$ | $\pi^n$ |
|---|---|---|---|---|---|---|
| $w(x)$ | $0$ | $\pi$ | $2\pi$ | $3\pi$ | $\cdots$ | $(n-1)\pi$ |
| $f(x)$ | $0$ | $0$ | $0$ | $0$ | $\cdots$ | $0$ |

From the table, if $x = \pi^n$, $w(x) = (n-1)\pi$.

Computing for $n$,

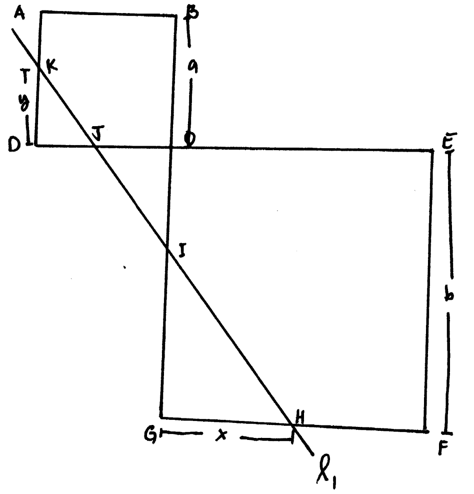
$$x = \pi^n$$
$$\log_\pi x = n$$

Substituting to $w(x)$,

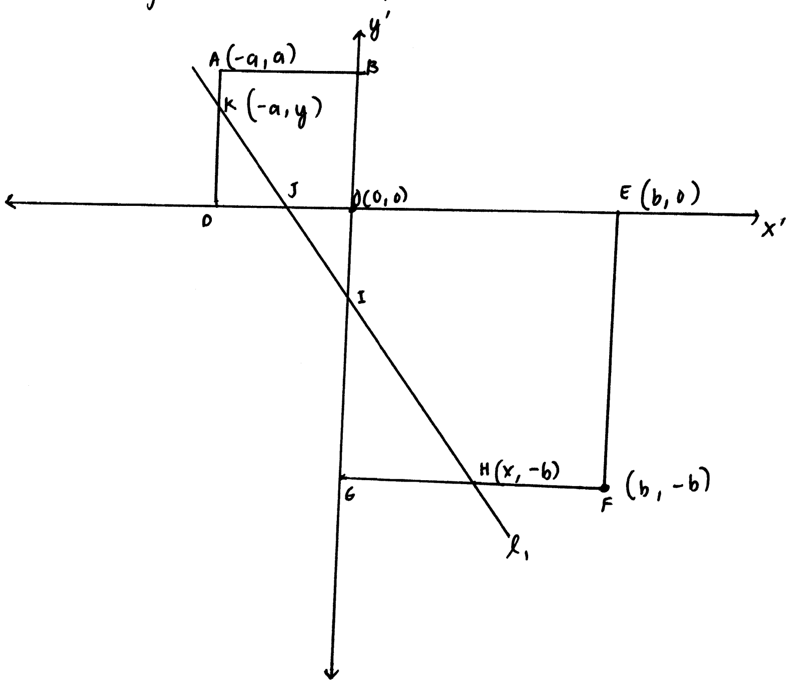
$$w(x) = (n-1)\pi$$
$$w(x) = (\log_\pi x - 1)\pi$$

Problem B.6.

Reconstructing the figure and labelling points,




Assuming that $\overline{AB} \parallel \overline{DE} \parallel \overline{GF}$ and O is the common vertex of the two squares. Placing the diagram in an $x'y'$- coordinate system (to avoid confusion with the given lengths x & y).



Points H, I, J, and K all lie on the line $l_1$. Therefore the line that describes $\overline{HK}$ is also the line that describes $\overline{IJ}$. Since H and K are expressed in terms of $a, x, b, \& y$, the points that will be produced will also be in those variables. Solving for the equation of $l_1$,

$$y' - y = \left(\frac{-b-y}{x+a}\right)(x'+a)$$

The $y'$-coordinate of I is,

$$y' - y = \left(\frac{-b-y}{x+a}\right)(0+a) \qquad \text{So} \quad I\left(0,\ a\left(\frac{-b-y}{x+a}\right)+y\right)$$

$$y' = a\left(\frac{-b-y}{x+a}\right) + y$$

The $x'$-coordinate of J is

$$0 - y = \left(\frac{-b-y}{x+a}\right)(x'+a)$$

$$-y = \left(\frac{-b-y}{x+a}\right)x' + a\left(\frac{-b-y}{x+a}\right) \qquad \text{So} \quad J\left(\frac{y(x+a)}{b+y} - a,\ 0\right)$$

$$x' = \left(\frac{x+a}{-b-y}\right)\left(-y - a\left(\frac{-b-y}{x+a}\right)\right)$$

$$= \frac{y(x+a)}{b+y} - a$$

The area in question, area$(OJI)$ is given by

$$\text{area}(OJI) = \frac{1}{2}\cdot OJ \cdot OI$$

since $OJI$ is a triangle with base $\overline{OJ}$ and height $\overline{OI}$

Calculating the lengths of $\overline{OI}$ and $\overline{OJ}$,

$$OI = \sqrt{0 + \left(a\left(\frac{-b-y}{x+a}\right) + y\right)^2} = \left|a\left(\frac{-b-y}{x+a}\right) + y\right|$$

$$OJ = \sqrt{\left(\frac{y(x+a)}{b+y} - a\right)^2 + 0} = \left|\frac{y(x+a)}{b+y} - a\right|$$

$\therefore$ the area OJI or $A(a,b,x,y)$ is

$$A(a,b,x,y) = \frac{1}{2}\left|a\left(\frac{-b-y}{x+a}\right) + y\right|\left|\frac{y(x+a)}{b+y} - a\right|$$

$$= \frac{1}{2}\left|\frac{-a(b+y) + y(x+a)}{x+a}\right|\left|\frac{y(x+a) - a(b+y)}{b+y}\right|$$

$$= \frac{1}{2}\left|\frac{(xy-ab)^2}{(x+a)(b+y)}\right|$$

Since the numerator is always positive and length $a, b, x, y$ are also always positive,

$$A(a,b,x,y) = \frac{(xy-ab)^2}{2(x+a)(b+y)}$$